Using CAS to Enrich the Teaching and Learning of Mathematics

Robyn Pierce
r.pierce@ballarat.edu.au
University of Ballarat

Abstract

Computer Algebra Systems (CAS) are powerful tools for both doing and learning mathematics. They may be used to perform algorithmic routines both quickly and correctly but harnessing this power in a manner which is effective for promoting learning is not trivial. Research undertaken with both secondary school and undergraduate students clearly indicates that, while students quickly come to appreciate the availability of CAS to check their answers, several key factors influence the development of their use of the facility of CAS to extend both access to mathematics and support learning of mathematical concepts. First, the institutional value which the technology is afforded influences the degree to which students are willing to apply themselves to the task of learning technical skills necessary to work with CAS. Second, the use of multiple representations may both increase students' conceptual understanding and provide them with alternative methods through which to progress solution of problems. Finally, students need to be guided in judicious use of CAS. This will involve teaching students to be discriminating in their use of technology for functional purposes, that is, to find solutions to difficult or time consuming problems, and strategic in their use of CAS to explore patterns and link representations in order to gain greater insight into mathematical processes and concepts.

Introduction

Computer Algebra Systems (CAS) have been available for some decades but only as this technology has become both accessible to students and more affordable has it offered serious possibilities as a tool to enrich the teaching and learning of mathematics. This paper addresses not only the value which CAS may add to a mathematics classroom but also draws attention to the initial cost, in terms of time and effort, involved in learning to use both the hardware and the software. There is discussion of the mathematical thinking and technical skills which students require in order to monitor their working and access the power of CAS. Then, aspects and elements of *effective use of CAS* are summarised in a framework and the importance of both the technical and personal aspects explained. The section below begins with consideration of the potential benefits of CAS in the mathematics classroom.

Computer Algebra Systems: what do they offer mathematics teachers and students?

Computer Algebra Systems (CAS) are powerful tools for both doing and learning mathematics. Both teachers and students may appreciate the possibility of sharing cognitive load with such technology. While we could sit down and do by-hand the type of arithmetic sums my grandfather referred to as 'long tots', or work out by-hand the class mean and standard deviation for a set of tests results, most of us prefer to save time and rely on calculators or spreadsheets instead. In the same way CAS may be used to perform a range of algorithmic routines both quickly and correctly. Students often feel that, not only will they obtain an answer more quickly than if they wrote out the necessary steps by-hand, but that they are less likely to make an error such as an incorrect sign or index. Kissane (2000) refers to graphic calculators and CAS as 'enabling technology'.

This concurs with the views of a class of Australian senior secondary school students (17 year olds) who were participants in a research project which monitored the introduction of a new mathematics subject permitting CAS use in any aspect of the course and its assessment. (HREF1). These students were regularly surveyed and interviewed in both Years 11 and 12. Typical survey responses from these students included the following:

Student 11A	Great to work out answers, gives me an idea of what to expect of the answers. Very helpful in graphs and finding out things.
Student 11B	I love using CAS and [it] makes me feel confident of the work I have done. I feel advantaged about using it as it teaches things that otherwise I may not have understood.
Student 12A	I prefer pen and paper and only use CAS if I know it will solve the problem quicker or if I am checking the answer.
Student 12B	I enjoy using CAS as it makes maths more understandable and therefore more interesting.

These responses indicate that, while these students valued the availability of CAS to produce correct answers quickly, at least some students were also using CAS to support their learning at a deeper level. Böhm et al in their extensive paper, prepared for T^3 World Wide, titled *The Case for CAS*, claim that:

> …provided students have access to CAS technology at home and in the classroom, effective use of CAS can have a number of important benefits. These include:
>
> - Making concepts easier to teach because students can approach situations numerically, graphically and symbolically,
> - Supporting visualisation and allowing situations to be explored that cannot otherwise be, thus enabling students to take mathematics to a more advanced level,
> - Saving time on routine calculations - time that can be exploited to study effects, outcomes and 'what if….?' situations,
> - Improving the students' perception of mathematics and consequently increasing their enjoyment of the subject and their motivation to learn.

CAS may be used to allow students to explore mathematical patterns: both arithmetic and symbolic, link representations of functions: numeric, symbolic and graphic and to tackle non-sanitised real world problems. CAS sometimes generates results which are unexpected or not expressed in the same way that we have come to regard as conventional, when working by-hand. These idiosyncrasies may be used as the catalyst for rich mathematical discussion; enhancing students' algebraic understanding through discussion of equivalent expressions or resolution of cognitive conflict.

The outputs from CAS add a new variable to the classroom. They may provide feedback, illustrations and stimulus to mathematical thinking previously primarily provided by the teacher and the text book. The availability of CAS for each student in a classroom will most likely change the classroom dynamics. Guin and Trouche (1999) describe the CAS classrooms they observed as follows:

> Throughout the lesson both a blackboard and a screen (displaying one of the calculators) were used. This combination enabled the individual student's work, both on paper and using the calculator to be guided by the teacher… each student took a turn operating the projected calculator. This student, called the 'sherpa student' played a central role in the layout of the lesson as a guide, assistant and mediator. Traditional classroom relations were altered: new classroom relations were established between the sherpa students and the other students as well as between the sherpa student and the teacher. This new context favoured classroom debates, pointed out student behaviours, and was essential to counterbalancing the rather individualistic relationship students tend to have with the small screen (p209).

Garner (2004) reports that when she started working with CAS, the dynamics of her classroom changed with the calculator viewscreen taking a central focus. For her, an "unexpected by-product of the viewscreen in the classroom at all times is the ease with which students can demonstrate their working to others". Garner shares access to the projected calculator with her students. Students usually follow their teacher's example but, as the student comment below indicates, not always…

Student 11D	I generally follow the way [our teacher] uses CAS, [but] if I know a quicker way or one that I understand more I use that in preference.

The availability of CAS can empower students to experiment, follow different solution methods and explore variations or different mathematics.

Overall, researchers agree that CAS offers many possibilities but they have also found that without further appropriate guidance students tend to limit their use of CAS to checking results or simply doing required calculations quickly, they may fail to gain any advantage of the availability of CAS to extend their mathematical applications and their mathematical understanding (see for example: Guin & Trouche, 1999, Pierce, Herbert & Giri, 2004, Pierce & Stacey, 2004a). As we will now discuss, learning to harness the power of CAS in a manner which is effective for promoting learning as well as doing mathematics is not trivial.

Learning to use CAS, as a teacher or a student, is not trivial

Our senior students indicated that CAS use is not always straight forward and can sometimes get in the way of the mathematics they need to focus on. Students 11C, 12C and 12D provide typical responses:

Student 11C	It's great for solving problems, though often I forget how to type it [mathematical expressions] into CAS. Sometimes it gives a complicated answer and to get that I have to type it in somewhere else.
Student 12C	CAS can either be really helpful or so annoying. There are too many types of errors and I have no idea what they all mean.
Student 12D	Most of the time it is good, but sometimes it is hard to work out what you want to do and how to get the answer you need. Sometimes [it seems] too complicated.

Artigue (2002), reflecting on a major French study of the use of CAS for secondary school mathematics, refers to the "unexpected complexity of instrumental genesis (ie. of the process of becoming a skilled user)". She comments that the complexity of the process "does not fit visions that consider technology mainly as an easy tool for introducing students to mathematical contents and norms defined independently from it" (p253).

This complexity is illustrated by Figure 1 (Pierce & Stacey, 2004a) which indicates a continuum of issues extending from learning the features of hardware, through to issues of mathematical thinking with or without the interface of technology. The breadth of *technical aspects* indicates that there are many issues related to the knowledge and skills required to work at the interface between the machine and the mathematics.

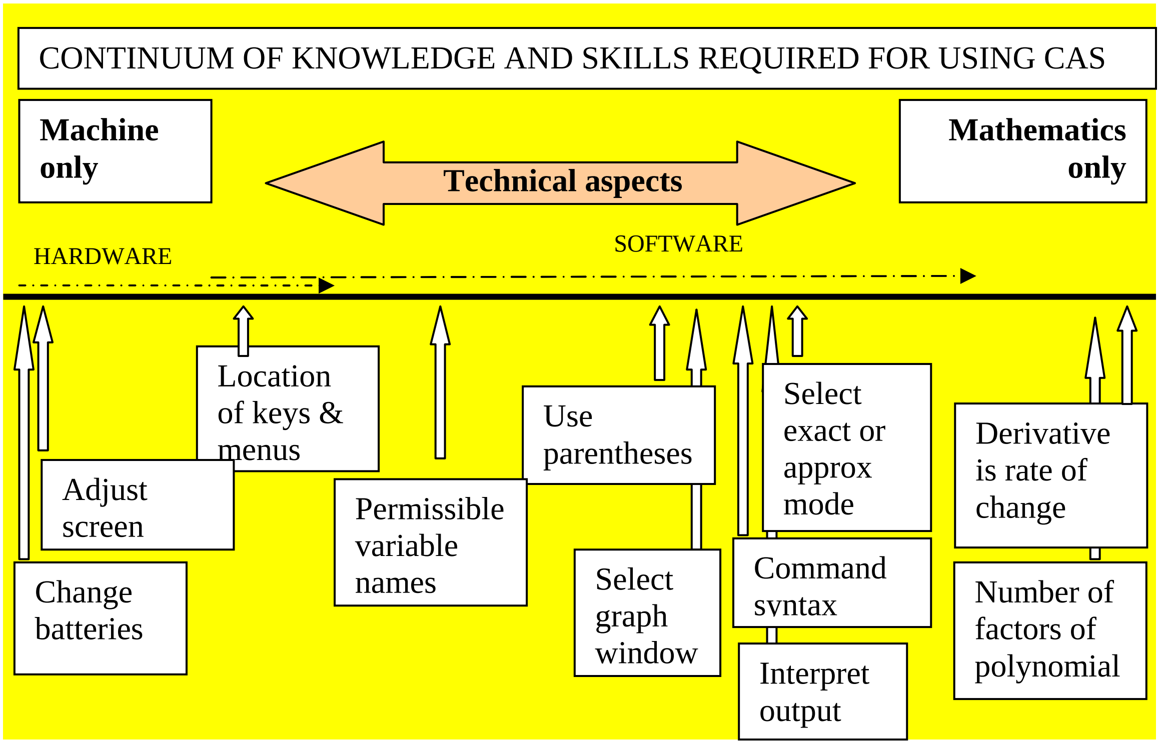

Figure 1. The continuum of knowledge and skills required for using CAS (Pierce & Stacey, 2004a)

Dealing with the machine: learning the basics of hardware and software

At the most basic level, CAS users must be able to turn the CAS on and off, change the batteries, adjust the screen and locate appropriate keys and menus. These skills are directly related to the hardware of the machine. While locating a particular key is a hardware issue at one level, it may also be a software issue. Knowing the possibilities offered by the software includes knowing that single keys may have several purposes if used in the correct combination or in relation to specific menus. No doubt the physical design of machines will continue to improve, but for current CAS, not all layout features or menu command structures, or names of commands are intuitively obvious in either their location or their meaning. Successful use of CAS relies on the user undertaking sufficient practice to become fluent with syntax, keystrokes and menus. Such practice requires motivation: a sense of need or purpose which will make the effort required worthwhile.

Clearly there is, just at the machine level, an overhead in learning to use any technology whether it be, for example, a spreadsheet, function grapher or CAS. The cost, in terms of time and money, must be weighed against its potential or planned use.

If the students are going to make good use of CAS then the investment made in learning the features of hardware and software pays off but to be valued the level of machine learning undertaken at each stage should be of immediate necessity to students.

Progressing to the right on the continuum illustrated in Figure 1 we see a range of *technical issues* which will be encountered by the CAS user. These involve not only issues of syntax, another CAS overhead, but also mathematical questions which arise as the user interprets CAS output.

Working at the interface between software and mathematics requires technical know how and algebraic expectation

One challenge for new CAS users is becoming familiar with the precision of CAS syntax necessary to correctly enter mathematical expressions into the machine. The most common problems arise with the use of parentheses which are often required in order to make the structure of expressions explicit. To make appropriate decisions about the placement of parentheses, the user must be able to correctly parse algebraic expressions, identifying structure and groups of components which should be treated as objects. The use of parentheses is a common source of errors which students must be able to recognise and then correct.

Checking for correct entry of mathematical expressions is the first stage in monitoring mathematical work with CAS, a process which requires *algebraic expectation*, a notion for algebra which parallels the skill of estimation for arithmetic. This concept is detailed by Pierce & Stacey (2004b) where it is described as the first aspect of *algebraic insight*. (A summary framework is included in this paper as Appendix 1). At all times the CAS user must be alert to errors, checking the expressions on the screen against the structure and features they might *expect* to see given the nature of the mathematical task.

The first element of algebraic expectation is *recognition of conventions and basic processes*. This includes common instances such as knowing the meaning of symbols and recognising implicit multiplication. This knowledge can be important for students assigning variable names or interpreting the errors which occur when multiplication is not made explicit. The syntax required by CAS has features of classic programming as well as conventional mathematics. The specific details will vary from program to program and between calculator brands but Figure 1 notes ‘permissible variable names’ and ‘command syntax’ as features of software which we have seen cause problems for students. CAS allows assignment of values or expressions to specific letters or ‘words’ in the same way as computer programs typically allow assignment of variables. The use of variable names and remembering when values have been assigned to specific letters may be a new skill for students being introduced to CAS.

When working by-hand, students would be familiar with writing a formula using ‘words’ but then representing those variables with single letters when performing calculations. The designation of groups of letters as variable names may cause confusion. Consider the task of factorising the trinomial $x^2 - 4xy - 21y^2$, illustrated in Figure 2a: a student may reasonably expect that since their TI-89 calculator factorises $x^2 - 4x - 21$ as $(x-7)(x+3)$ they need only edit this expression to insert the *y*’s. The unexpected CAS response, illustrated in Figure 2a, occurs because the *xy* multiplication must be made explicit, $x * y$; *xy* may be used as a variable name and the machine cannot discriminate between these options.

CAS users need to be able to identify keying or syntax errors and when to make structure and processes explicit through the use of parentheses and a multiplication symbol. They also need to be able to recognise equivalent expressions which may be displayed as a result of the software's automatic simplification processes. For example, entering the simple expression *ab-mn+an-bm,* on a Texas Instrument's TI-89 calculator, unexpectedly produces a semi-factorised simplification as shown in Figure 2b. It is common for students to believe that the CAS will automatically perform all the manipulations they will require but this is not always so. To work with the outcomes of CAS processes, students need to have an expectation of likely mathematical results. They must decide whether the calculator response is: reasonable; equivalent; complete; or helpful for progressing towards the solution of a particular problem.

Recognising equivalent expressions and dealing with such unexpected outcomes requires a well developed sense of algebraic expectation, in particular the ability to *identify structure* and *key features*, for example: recognise simple factors, trigonometric identities or the highest power of polynomial. Using CAS to work with trigonometric expressions, requires the student to be familiar with a range of trigonometric identities. In Figure 2c the expression $\tan(x)+\sin(x)$ is returned with an automatic simplification to $\tan(x)(\cos(x)+1)$ and the expansion of $(\cos x+\sin x)^2$ does not return the full expansion $\cos^2 x+2\sin x\cos x+\sin^2 x$, rather $\cos^2 x+\sin^2 x$ is automatically replaced by 1. The results of automatic simplification by CAS may produce results which are quite unexpected for the student. Such incidents may be the catalyst for valuable mathematics discussions which would not normally arise when students work only with pen and paper.

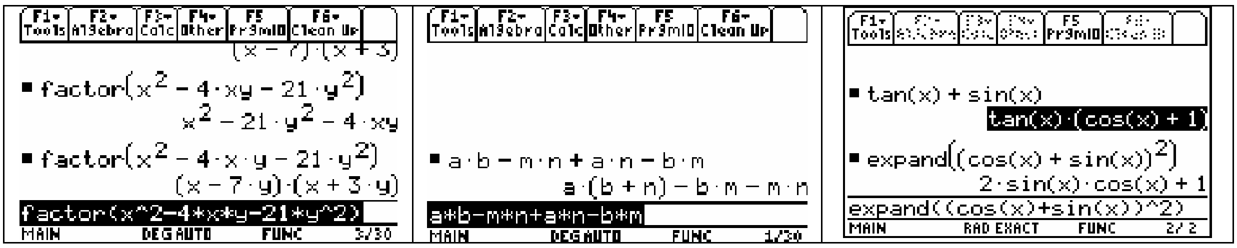

Figure 2a, 2b &2c Unexpected equivalent expressions

Realising the potential of CAS for doing and learning mathematics

The availability of technology, such as CAS, adds to the options we, as teachers, have available to empower our students to support and extend their facility with, and understanding of, mathematics. However, as discussed above there is much to learn. Successful use of CAS requires the user to gain technical skills in order to efficiently operate the machine and to develop algebraic insight in order to monitor their work and deal appropriately with unexpected outcomes. If CAS is to add value to a classroom, teachers need to develop specific skill in teaching mathematics with technology. Thomas and Hong (2005) comment that it is now clear that knowing how to control the functioning of a tool such as a graphics calculator or CAS, is insufficient for a successful mathematical outcome; teachers need to develop *pedagogical technology knowledge*. Teachers, as well as students, need to go through the processes of *instrumentation* and *instrumentalization* (Artigue,2002), that is, not only learning to efficiently use the tool (CAS) to do mathematics but also considering how conventional solution processes may be changed, improved or alternative methods employed because the technology is available. Some teachers use CAS as an integral part of their teaching and working in mathematics while others treat it as an 'add-on'.

Kendal and Stacey (2001) describes teacher privileging of different representations of mathematical ideas available with CAS. They indicate that students' use of CAS may be expected to follow quite closely that modelled and demonstrably valued by the teacher. In the section below, we consider features of teaching with CAS that are most likely to allow this facility to enrich the learning of mathematics.

Research undertaken with both secondary school and undergraduate students indicates that, while most students quickly come to appreciate the availability of CAS to check their answers, several key factors influence the development of their use of CAS to both extend access to mathematics and to support their learning of mathematical concepts. Pierce & Stacey (2004a) discusses the aspects and elements of, what we call, Effective Use of CAS. This framework (summarised in Table 1) suggests that there are two key aspects of Effective Use of CAS, *technical* and *personal*. The first aspect, *technical*, refers to the skills and thinking required to access the potential of CAS. This aspect focuses on the interface between the machine and the mathematics as illustrated by Figure 1. The second aspect, *personal*, and the *technical* aspect are not disjoint; time and again we have observed clear evidence of the impact of teachers' and students' attitudes on their development of technical skills. This is illustrated in the first section below which discusses the impact of the institutional value which is given to the use of CAS. In the second section, one of the most commonly noted potential affordances of CAS, the ability to work with multiple representations, is discussed. Finally, the third section expands on the importance of assisting students to learn to make judicious use of CAS, that is, to be discriminating in their functional use and strategic in their pedagogical use of this instrument.

Table 1
Effective Use of CAS Framework (Pierce & Stacey, 2004a)

Aspects		Elements		Common Instances	
1.	Technical	1.1	Fluent use of program syntax	1.1.1	Enter syntax correctly
				1.1.2	Use a sequence of commands and menus proficiently
		1.2	Ability to systematically change representation.	1.2.1	CAS plot a graph from a rule and vice versa
				1.2.2	CAS plot a graph from a table and vice versa
				1.2.3	Create table from a rule or vice versa
		1.3	Ability to interpret CAS output	1.3.1	Locate required results
				1.3.2	Interpret symbolic CAS output as conventional mathematics
				1.3.3	Sketch graphs from CAS plots
2.	Personal	2.1	Positive attitude	2.1.1	Value CAS availability for doing mathematics
				2.1.2	Value CAS availability for learning mathematics
		2.2	Judicious Use of CAS	2.2.1	Use CAS in a strategic manner
				2.2.2	Discriminate in functional use of CAS
				2.2.3	Undertake pedagogical use of CAS

Institutional value encourages students to conquer technical overhead

Affording CAS institutional value, ie. official acceptance, through discussing and modelling appropriate use in the classroom and allowing its use for assessment tasks will help to foster a positive attitude towards the use of CAS. A positive attitude is required if students are to apply themselves to the task of gaining technical skills.

Some weaker students need little convincing to adopt the use CAS; they almost immediately see it as valuable for supporting their learning of mathematics. Students such as student 12E welcome the availability of CAS and are willing to make the investment necessary to learn technical skills.

> Student 12E Without CAS, I would be lost, it helps me understand maths in general. With CAS, I CAN DO MATHS CONFIDENTLY!

Other, often more mathematically confident students, like student 12F, may be anxious about their technical facility or even resentful that a machine can do what they have struggled to learn.

> Student 12F I like the fact that I can fully understand some of the concepts of a lot of the harder maths, but sometimes I don't like putting my full trust [in CAS] in case I do something wrong. [I find] it's hard to accept the fact that a lot of things that would have to have been done by-hand can now be done instantly…

Such students seem to feel that CAS devalues their ability to perform routine manipulations by-hand. While perhaps mathematically able, they sometimes fail to find advantage in CAS because they do not apply themselves to the necessary task of acquiring technical competence. Despite his hesitancies, Student 12F acquired CAS skills and appreciated its role as a learning tool. The impetus for him to conquer the technical obstacles came from the official acceptance of CAS in his course: its use was expected in the high-stakes final examinations.

Such institutional value strongly influences the degree to which students are willing to apply themselves to the task of learning the technical skills, and new approaches to mathematics, necessary to work with CAS. Institutional value is not indicated by merely making CAS available for students. Its acceptance is credible when CAS is valued for both functional and pedagogical purposes as well as permitted for assessment tasks, including examinations. Students will be more likely to see CAS as valuable if their teachers encourage its use and consistently model both *when* and *how* to use it effectively.

Even official acceptance and status may not be sufficient to prompt students to use CAS if this status is not mirrored in the teachers' actions. Pierce, Herbert and Giri (2004) report on a 2003 study of tertiary students studying an introductory functions and calculus course, taught by two different teachers. One teacher had a favourable attitude towards the use of CAS for learning mathematics but was assigned to a classroom unsuited to displaying the viewscreen image. As a consequence, he had little opportunity to provide of model of CAS use for his students. The second teacher was openly negative about the facility offered by this technology. He expressed concern about limited screen resolution and placed emphasis on working by-hand. He discouraged CAS use to the extent that two of his students returned their calculators half way through the course. Despite the fact that CAS was allowed in the examination, analysis of the students' examination scripts suggested that only about one third of the students had actually used their CAS.

Students recorded incorrect answers to problems such as simple derivatives or integrals which require only trivial use of CAS. There was no evidence that they had even checked their answers. If students are to be encouraged to make effective use of CAS then teachers' affirmation of its use must be unambiguous. They need to give clear guidelines for its use and if students use CAS as an integral part of their learning, then it must also be an integral part of the assessment.

Multiple representation aids understanding and offers options

One of the approaches to learning mathematics which can be encouraged with CAS is that of versatile thinking: for example, exploring mathematical expressions and the solution of problems through the use of symbolic, graphic or numeric representations. The use of multiple representations may both increase students' conceptual understanding and provide them with alternative methods through which to progress solution of problems. Many researchers (see for example: Tall and Thomas, 1991) have discussed the contribution of experience gained from working with different representations of a problem (for example: symbolic, graphic and numeric representations of functions) to a student's development of strong cognitive schema.

CAS allows students to move easily between these representations. For example, the function $f(x) = x^3 + x^2 - 6x$ may be explored in some of the ways shown in Figure 3. The function defined in the symbolic module may be graphed in the graphic module, the critical values may be found approximately in this module or exactly using conventional algebraic and calculus methods in the symbolic module. The tabular module may assist students to set a suitable graph window, locate zeros or consider successive difference scores. The combination of these representations enriches the student's understanding of this cubic function and offers a range of techniques from which they may choose in order to progress the solution of various problems.

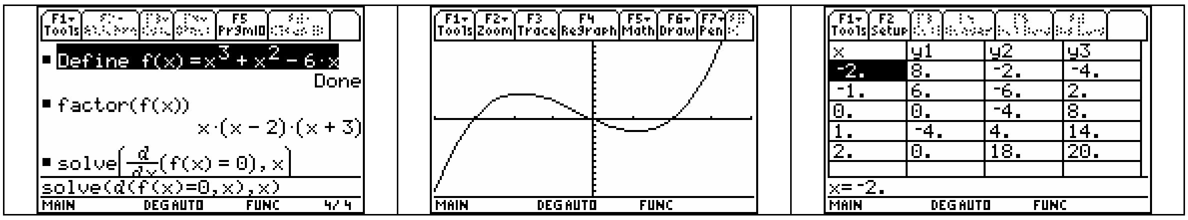

Figure 3 Example showing three representations of a cubic function.

Working with multiple representations does require the user to learn the technical skills associated with moving between representations. In particular, CAS are typically programmed to graph functions where *y* is expressed as a function of *x* and so for functions of other variables, a substitution is necessary. Understanding this process requires algebraic insight (see Appendix 1): in this case understanding the meaning of letters used as variables.

Ability to link representations is the second aspect of algebraic insight. Awareness of the possible form and key features alerts the student to obvious problems like typographical errors and assists with setting a suitable graph window. In the example above, for instance, if the student identified the key feature of an *x* in each term and linked this to the fact that the graph of this function should pass through the point (0,0), then the graphical image on the left in Figure 4 would alert the student to an error.

On the other hand, considering the numeric representation of the correct function, shown in the table on the right hand side of Figure 4, may assist the student in setting the bounds of a suitable graph window.

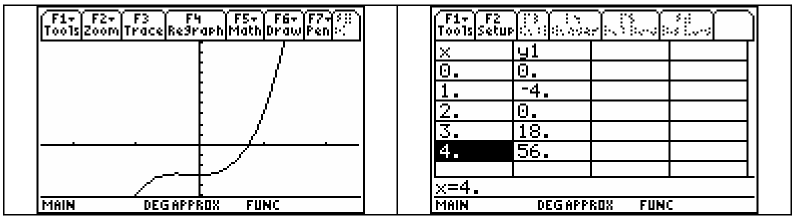

Figure 4 Algebraic Insight: identifies errors and assists with window setting

The exciting 'explosion of available solution methods' (Artigue, 2002) presents several challenges for teachers. Some methods may be unexpected, taking the teacher by surprise and requiring some quick thinking on their part, but, more importantly, students may require guidance in their choice of methods or encouragement to explore more than one method. In some cases, an approach may be adequate for a limited number of examples but not provide a generally applicable solution method, in other cases teachers may wish to privilege a particular method in order to open up some new aspect of mathematics to the students. Some researchers (for example Artigue, 2002) believe that an 'explosion-reduction' is necessary. They would encourage a teacher to 'institutionalise' the methods which they consider to be of most value. This would be achieved by rewarding, or only accepting work which follows these designated methods. Artigue believes that it is important that further research effort should be put into establishing guidance for such 'best' practice.

Students need guidance to develop judicious use of CAS

Finally, students need to be guided to develop *judicious use* of CAS. This is the second element of the personal aspect of Effective Use of CAS as summarised in Table 1. It will involve teaching students to be discriminating in their use of technology for functional purposes, that is to find solutions to difficult or time consuming problems, and strategic in their use of CAS to explore patterns and link representations in order to gain greater insight into mathematical processes and concepts.

Students may use CAS as a 'partner' with 'whom' they share the cognitive load of doing mathematics but the value of this partnership will only be realised if the student uses CAS in a discriminating manner. As stated earlier, students value CAS for finding answers quickly and assisting with manipulations that they see as 'hard' or laborious and error prone. However, students must learn not to reach for the calculator in an unthinking manner. For many routine algebraic manipulations, especially at the secondary school level, if the student can identify the structure of the expressions, for example recognise simple factors, the quickest path to a solution may not involve using CAS. Students need to be aware that entering algebraic expressions using CAS syntax may be time consuming. If an expression is to be used repeatedly in the solution of a problem then this initial time spent on CAS entry may be worthwhile but students will require guidance in making these choices.

In addition to learning to make good choices about when to use or not use CAS for speed and convenience, students should be encouraged to make *strategic use* of CAS to support and extend their learning of mathematics. Pierce (2002) describes various approaches to the use of CAS. Some students have a *random* practice. They are typically overheard to say such things as "Oh, I just tried anything". Others prefer to wait for their peers to suggest a suitable tactic or they will only follow detailed sets of instructions. In contrast, strategic use requires the student to think about both the mathematics and the tool. In particular, to think about what they already know and understand about the mathematics under consideration and plan a systematic investigation, varying parameters and working between the available representations. As a result, students may identify patterns through a process of induction and then set about trying to prove the validity of any apparent generality. Students may be introduced to such an approach through structured experience with well designed tasks. Through strategic use, CAS may become a partner not only for doing mathematics but also for learning mathematics.

Conclusions

Research indicates that Computer Algebra Systems may be used as a valuable tool to support and promote the learning of mathematics. However when making the decision on whether or not to use CAS in a classroom the teacher must consider the initial investment of time and cognitive load associated with learning features of the machine and software. The value of CAS may be realised once working with the tool stimulates rather than impedes mathematical thinking. To reach this stage, students must be encouraged to develop the habits of discerning and strategic use of CAS along with the elements of algebraic insight required to monitor their work. The role of the teacher in modelling effective use of CAS and affirming its value through permitting its use for assessment tasks is vital in promoting the positive attitude needed to overcome any initial technical difficulties. Students working with CAS may: access multiple representations of functions; explore variations on set problems; observe the features of many correct solutions; or discuss the mathematics of equivalent expressions and unexpected output. In this way, CAS affords many opportunities for rich mathematical learning.

Acknowledgements

Thanks are extended to students who took part in various studies on the use of CAS at the University of Ballarat and to students and teachers who took part in the CAS-CAT Project, a major research project funded by the Australian Research Council, Casio, Hewlett-Packard and Texas Instruments.

References

Artigue, M. (2002). Learning Mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning 7*, 245-274.

Böhm, J., Forbes, I., Herweyers, G., Hugelshofer, R. & Schomacker, G. (2003). The Case for CAS. T^3 Europe Retrieved 8th September 2005 from http://www.t3ww.org/cas/index.html

Guin, D. & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: the case of calculators. *International Journal of Computers for Mathematical Learning 3*,195-227.

Garner, S. (2004). The CAS classroom. *Australian Senior Mathematics Journal 18*(11), 28-42

HREF1 CAS-CAT Project, University of Melbourne. Retrieved 8th September 2005 from http://extranet.edfac.unimelb.edu.au/DSME/CAS-CAT

Kendal, M. & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning 6*, 143-165.

Kissane, B. (2000). Technology and the curriculum: the case of the graphics calculator. In M.O.J.Thomas (Ed.) *Proceedings of TIME 2000: An International conference on Technology in Mathematics Education* (p p 60-71) Auckland NZ: The University of Auckland and Auckland University of Technology.

Pierce, R., Herbert, S., & Giri, J. (2004). CAS: Student engagement requires unambiguous advantages. In I. Putt, R. Faragher, and M.l McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010.* (Proceedings of the 27th annual conference of the Mathematics Education Group of Australasia, Townsville), pp.462-469. Sydney: MERGA.

Pierce, R. & Stacey. K. (2004a). A framework for monitoring progress and planning teaching towards effective use of computer algebra systems. *International Journal of Computers for Mathematical Learning. Vol 9*(1) 59-93.

Pierce, R.& Stacey K. (2004b) Monitoring Progress in Algebra in a CAS Active Context: Symbol Sense, Algebraic Insight and Algebraic Expectation. *International Journal for Technology in Mathematics Education. Vol* 11(1) 3-11.

Tall, D. & Thomas, M.O.J. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics 22*, 125-147.

Thomas, M.O.J. & Hong, Y. Y. (2005).Teacher factors in integration of graphic calculators in mathematical learning. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 257-264. Melbourne: PME.

Appendix 1

Framework for Algebraic Insight (Pierce & Stacey, 2002)

Aspects	**Elements**	**Common Instances***	
1. Algebraic Expectation	1.1 Recognition of conventions and basic properties	1.1.1	Know meaning of symbols
		1.1.2	Know order of operations
		1.1.3	Know properties of operations
	1.2 Identification of structure	1.2.1	Identify objects
		1.2.2	Identify strategic groups of components
		1.2.3	Recognise simple factors
	1.3 Identification of key features	1.3.1	Identify form
		1.3.2	Identify dominant term
		1.3.3	Link form with solution type
2. Ability to Link representations	2.1 Linking of symbolic and graphic representations	2.1.1	Link form with shape
		2.1.2	Link key features with likely position
		2.1.3	Link key features with intercepts and asymptotes
	2.2 Linking of symbolic and numeric representations	2.2.1	Link number patterns or type with form
		2.2.2	Link key features with suitable increment for table
		2.2.3	Link key features with critical intervals of table

* *Common instances* will relate to the age and stage of the students. Those selected here were based on an introductory functions and calculus course.