Using Chips to Understand the Sum of Progressions

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Abstract

It is very difficult for students to understand the formula $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$. Many textbooks use the expression $(n+1)^3 - n^3 = 3n^2 + 3n + 1$ as an explanation of the formula. However, it is too clever for many students to understand. This study describes one method to solve this problem based on arranging "chips" to construct squares, pentagonal numbers, cubes, etc. This way, the students gain an understanding of the mathematical constructions of these numbers. This paper describes a method to help the students gain insight into the sum of progressions as well as such formulas as $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$. The author wrote a Visual Basic program called "Chips" to help the students learn about the sum of progressions. Students construct chips on the computer screen to create the above numbers (e.g.., squares, cubes) as well as to find their regularity.

1. Materials

The following "chips" are prepared as a learning aid for the students. (Fig.1) Each chip consists of a rectangle inside of which we place k^l, where k (the base) and l (the index) are integers. The number 1 surrounded by the smallest rectangle represents k^0.

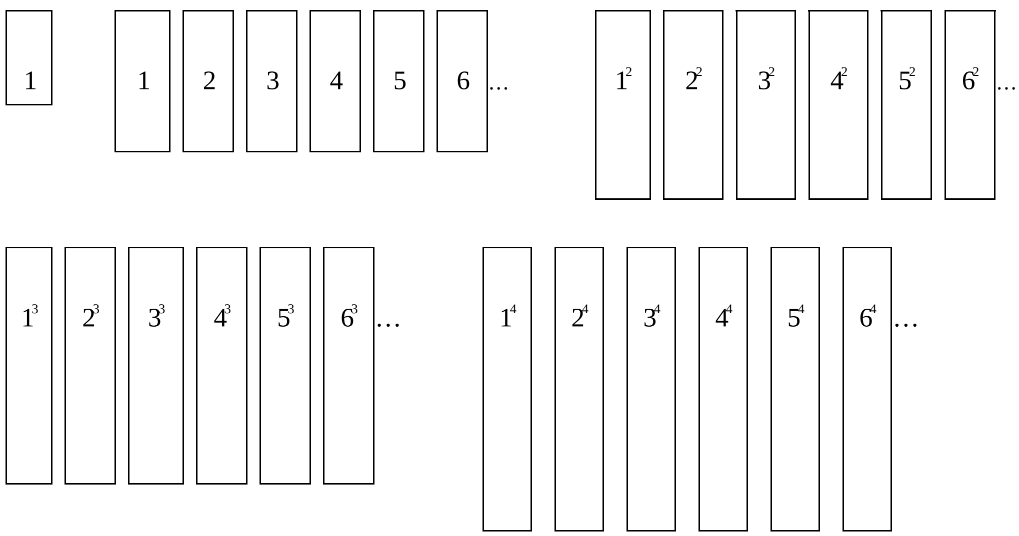

Fig. 1 Chips as a learning aid

The numbers $1,2,3,4\cdots$ surrounded by the next smallest rectangles represent k^1; the numbers $1^2,2^2,3^2,4^2\cdots$ surrounded by the following smallest rectangles represent k^2 and so on. As you can see, we distinguish between 4^1 and 2^2.

2. Constructing n^2 by arranging some chips

We set the base as 1, create some chips and line them up, as shown in Fig. 2. The partial sums of these chips are marked with dotted lines at the top, plus the base. The sum of all the chips is $(n+1)^2-1$.

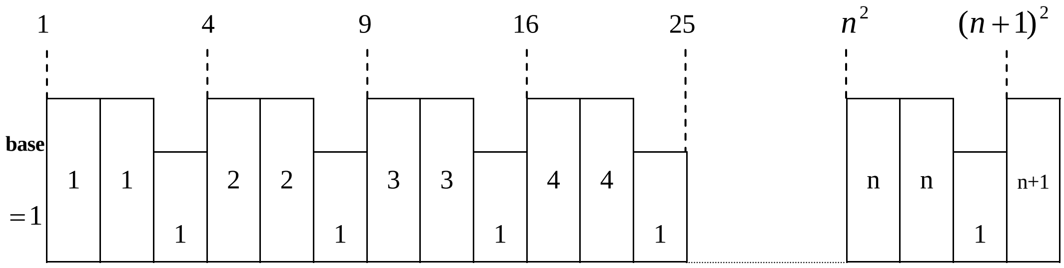

Fig. 2 Chips for the constructions of n^2. Squares are shown on the dotted line.

As you can see, we can construct squares by arranging two k^1 chips and one k^0 chip in order regularly. In fact, we can prove it through the expression $(n+1)^2-n^2=n+n+1$. By rearranging the same degree chips shown in Fig. 2, we can get the sum of $1+2+3+\cdots+n$.

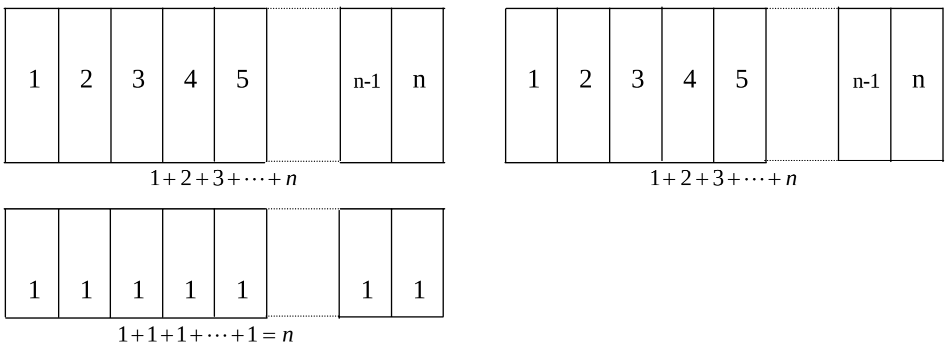

Fig. 3 Rearranging the same degree chips The total sum of all the chips is $(n+1)^2-1$.

We can see from Fig. 2 that the total sum of all the chips is $(n+1)^2-1$ and this value is equal to the total sum in Fig. 3. If we subtract $n=1+1+1+\cdots+1$ from the expression $(n+1)^2-1$, we can get two times of $1+2+3+\cdots+n$. If we divide this expression by 2, we can get the sum of $1+2+3+\cdots+n$. That is,

$$1+2+3+\cdots+n=\frac{(n+1)^2-1-n}{2}=\frac{n(n+1)}{2}$$

3. Constructing n^3 by arranging some chips

We can also construct n^3 by arranging chips as shown in Fig. 4. That is, three k^2 chips, three k^1 chips, and one k^0 chip. We can prove it through the following expression.

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

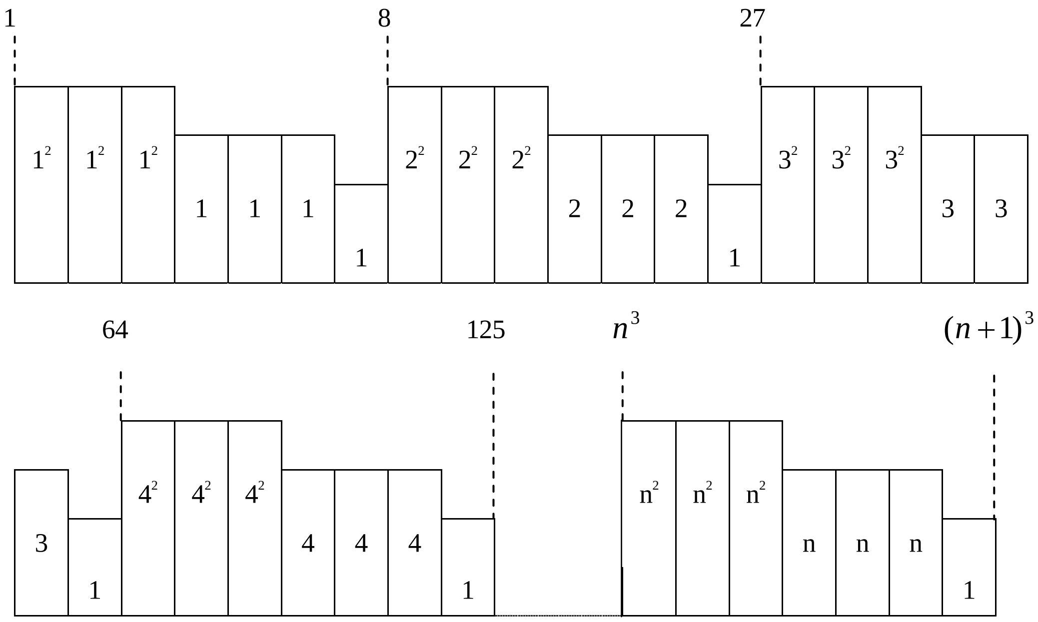

Fig. 4 Three k^2 chips, three k^1 chips and one k^0 chip are needed to construct n^3.

By rearranging the same degree chips, we get the following figure (Fig. 5)

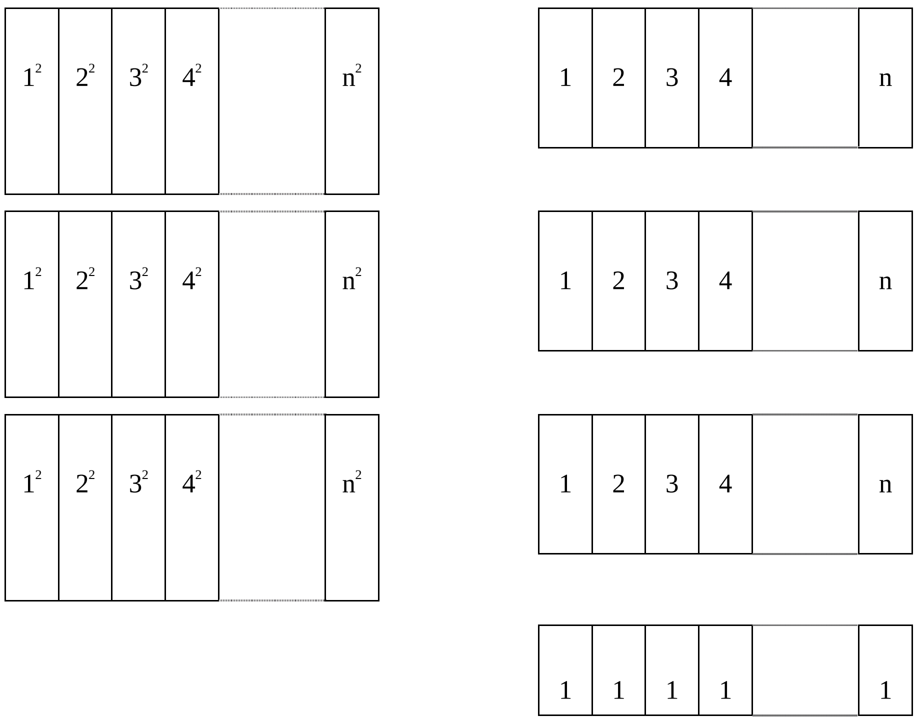

Fig. 5 Rearranging the same degree chips

Through Fig. 4, we can see that the total sum of all the chips in Fig. 5 is equal to $(n+1)^3-1$.
It turns out that this expression is divided into $3\sum_{k=1}^{n} k^2$, $3\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} 1$. In the same way we obtained the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ in section 2, we can calculate the expression $\sum_{k=1}^{n} k^2$ as follows.

$$(n+1)^3-1 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 + 3\cdot\frac{n(n+1)}{2} + n$$

$$\therefore \sum_{k=1}^{n} k^2 = \frac{1}{3}\left\{(n+1)^3 - 1 - 3\cdot\frac{n(n+1)}{2} - n\right\} = \frac{1}{6}n(n+1)(2n+1)$$

4. Constructing n^4 by arranging some chips

We can also construct n^4 by arranging some chips as shown in Fig. 6.

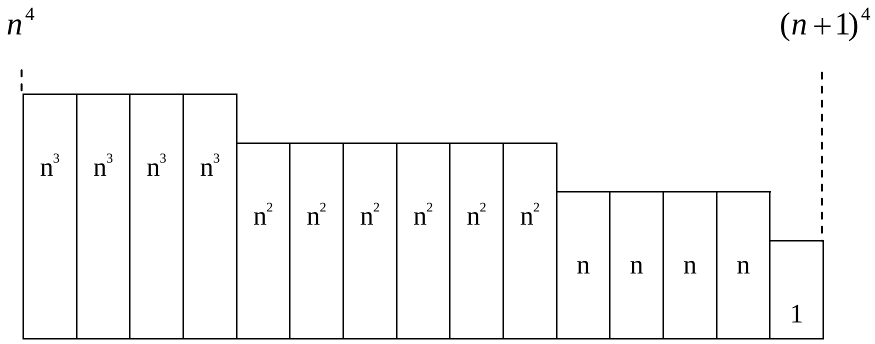

Fig. 6 Four k^3 chips, six k^2 chips, four k^1 chips and one k^0 are needed to construct n^4.

We can prove it through the expression $(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$. We can also calculate the formula $\sum_{k}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$ by rearranging the chips. The detailed explanation for this is omitted.

5. Constructing polygonal number by arranging some chips

Polygonal numbers consist of triangular numbers, square numbers, pentagonal numbers, hexagonal numbers and so on. [1)] We can also construct polygonal numbers by arranging some chips. For example, the nth pentagonal number is $\frac{1}{2}(3n^2-n)$ and we can construct this number by arranging some chips as shown in Fig. 8.

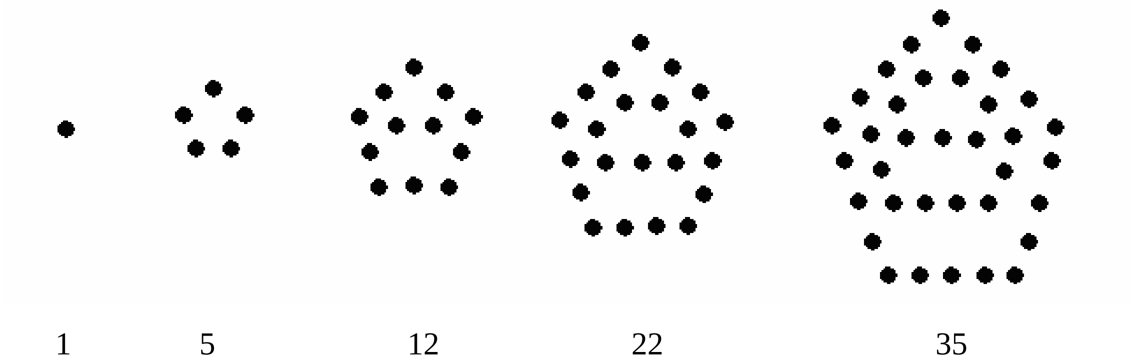

Fig. 7 The first five pentagonal numbers

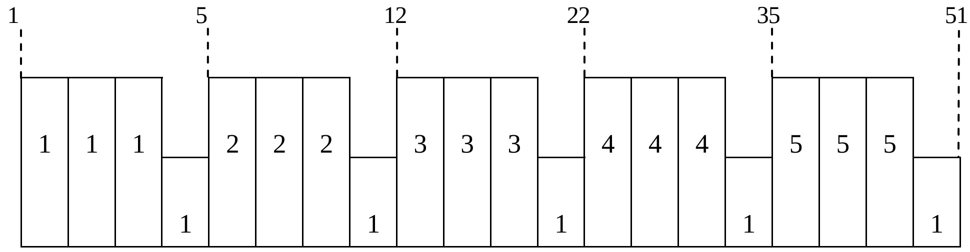

Fig. 8 Three k^1 chips and one k^0 chip are needed to construct the pentagonal numbers.

Hexagonal numbers also can be constructed as shown in Fig. 9 and Fig. 10.

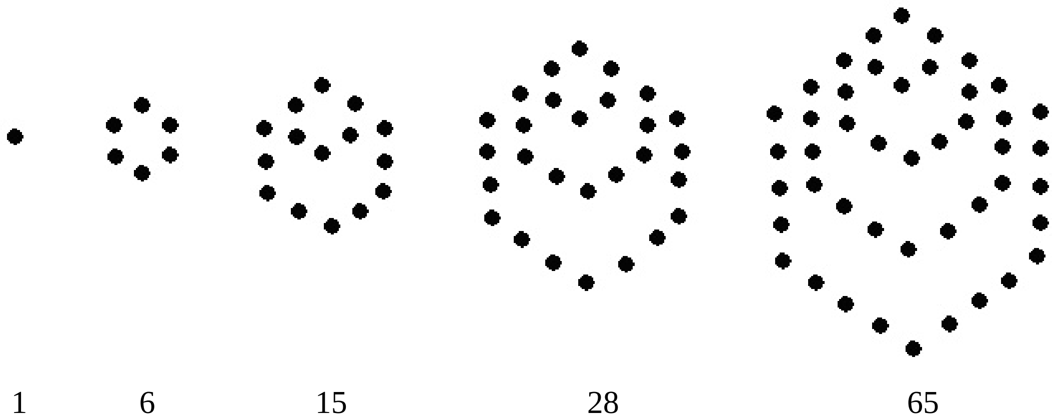

Fig. 9 The first five hexagonal number

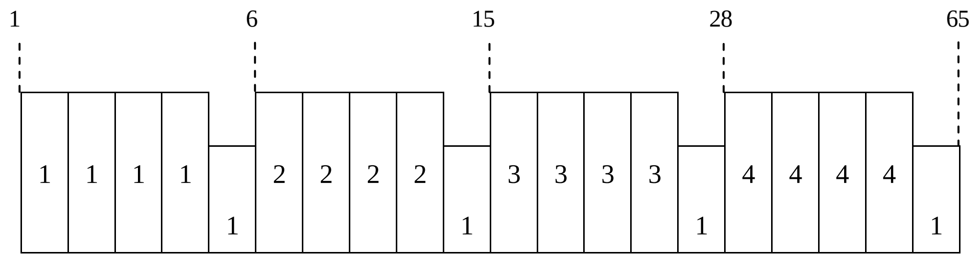

Fig. 10 Four k^1 chips and one k^0 chip are needed to construct the hexagonal numbers.

6. Constructing 3^n by arranging some chips

We can even construct 3^n by arranging some chips. If we arrange the chips as shown in Fig. 11, the total sum of all the chips plus base 1 is 3^n. This is proved by the expression $3^n - 3^{n-1} = 3^{n-1} + 3^{n-1}$.

Using this figure, we can calculate the formula $1 + 3 + 3^2 + \cdots + 3^{n-1}$. We can see that there are two sets of chips of 3^k and that we can divide them into the same two groups as shown in Fig. 12. The total sum of all the chips of the two groups equals $3^n - 1$. Therefore, the total sum of one group equals $\frac{3^n - 1}{2}$. That is, $1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$.

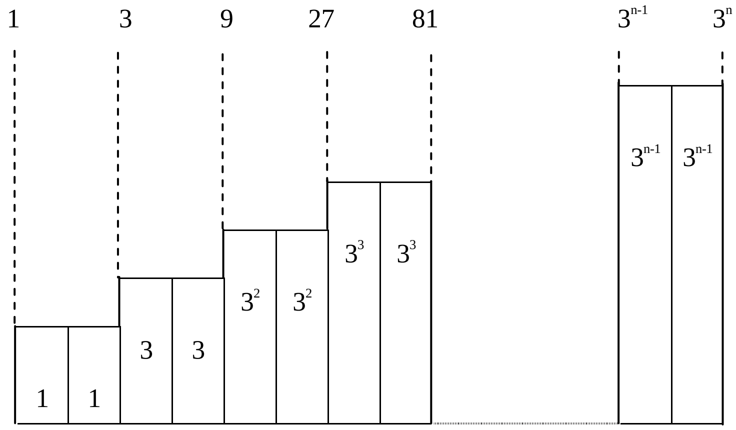

Fig. 11 3^n **is constructed using three sets of** 3^k**.**

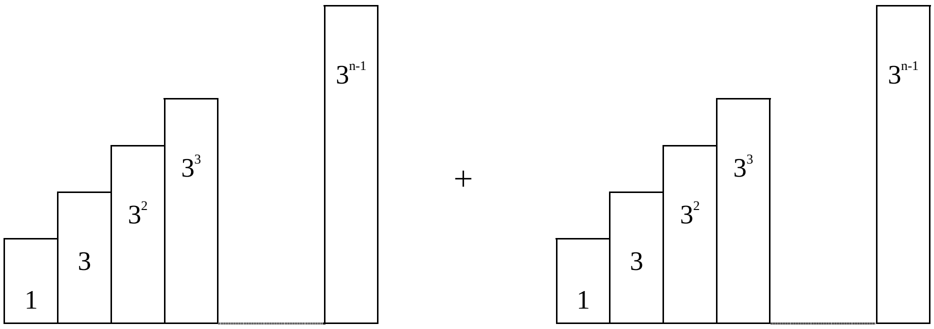

Fig. 12 Dividing the chips into two similar groups

We can similarly calculate the formula $1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}$ by arranging chips.

7. "Chips" ; A program for the sum of progressions

The author wrote a program called "Chips". This program can be used to help students understand the sum of progressions using chips on the screen. The first screen of this program is shown in

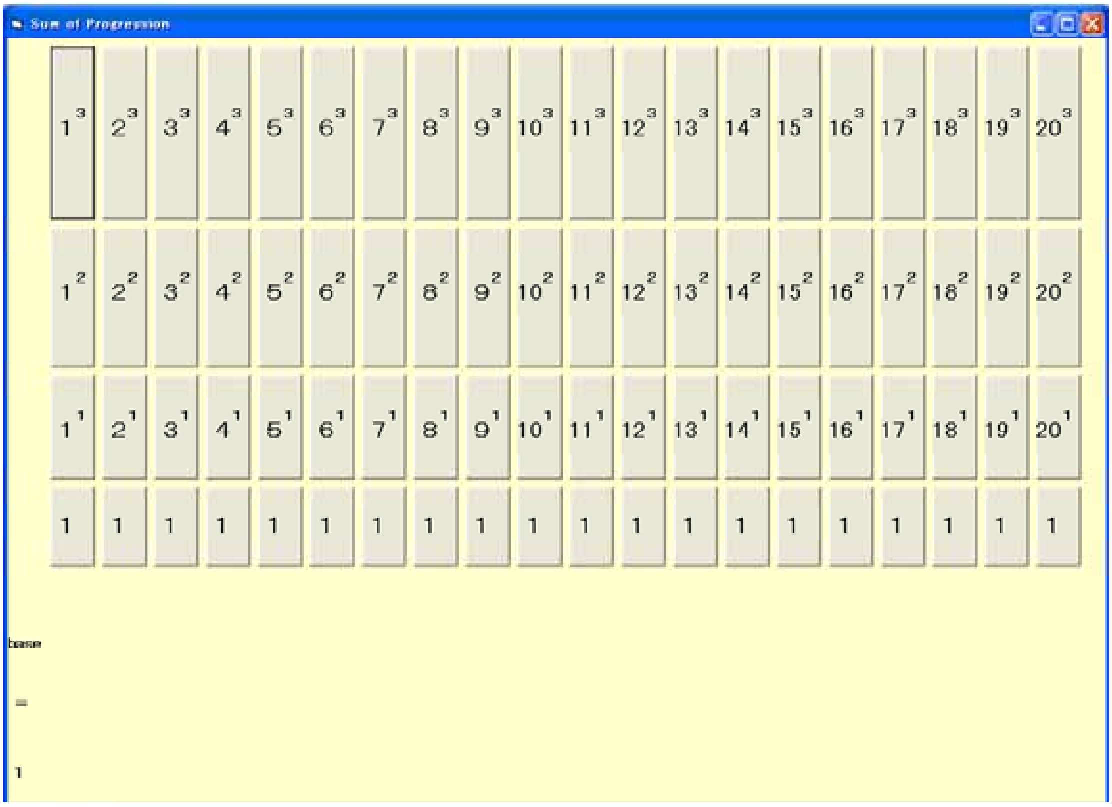

Fig. 13 The first screen of “Chips”

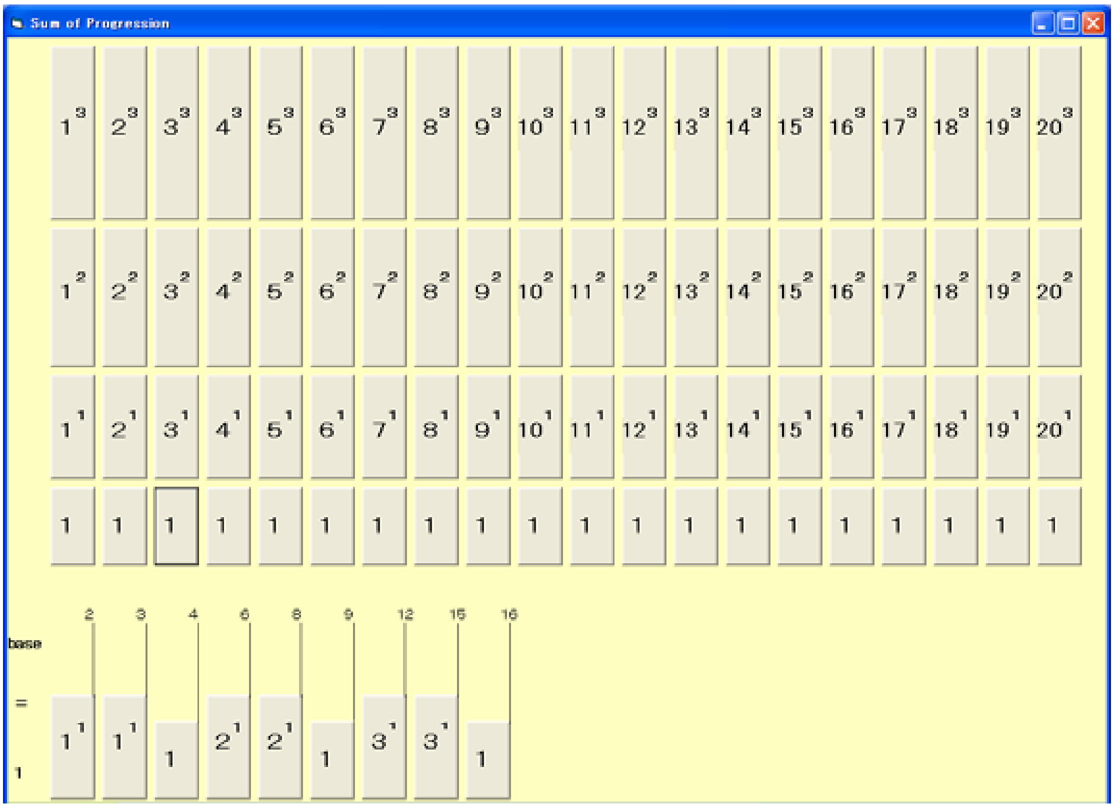

Fig. 14 Partial sums of the chips plus the base 1 are shown on the screen.

Fig. 13. Chips representing k^l are arranged in order. In this program, the base is set as 1 and if we move the mouse and click one chip, it is displayed at the bottom of the screen starting from the left. Fig. 14 shows the screen after clicking the chips $1^1, 1^1, 1^0, 2^1, 2^1, 2^0, 3^1, 3^1, 3^0$. The numbers on the lines represent the partial sums of the chips plus base 1. As you can see, some square numbers can be seen on the lines. By clicking chips on the screen, students can construct n^2 by trial and error. In a similar way, students will be able to construct other numbers such as $n^3, n^4, 3^n$, polygonal numbers, etc. Using this program, students can find insight into the mathematical construction of the sum of progressions.

8. Discussion

The author taught the sum of progressions using the "Chips" program to eleventh-grade students. They showed great interest in the constructing chips and were able to understand the sum of progressions. Many different impressions were expressed, as can be seen from the following examples.

· It is a good mathematical application for calculating the sum of progressions. It was a wonderful experience for me.

· When I first studied progressions, I thought they were difficult and boring. But this model taught me that many progressions can be constructed using some basic progression such as n^2

· I was surprised that n^m can be constructed by some progression, such as $1^k + 2^k + \cdots + n^k$ $(k = 0,1,2\cdots)$. It is a wonderful idea.

· I learned mathematics in order to solve the entrance examination test and considered mathematical knowledge only as a skill for solving problems. But this lesson has changed my way of thinking. Mathematics itself is interesting and accessible.

· It was an interesting lesson for me because we can obtain the formula of $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ clearly. When I first learned this formula, it was difficult for me to understand it.

· The program is good and if you sell it you will obtain profits. I had a great time with it!

· I am interested in computer methods. I would like to develop a program based on this idea in the near future.

· Calculating progressions was dull for me. But now, it has become meaningful thanks to this lesson. I want to learn more difficult sum of progressions.

I have written several practical educational programs based on computer assisted instruction to help high school students understand abstract mathematical concepts or applied mathematics.[2),3),4),5),6),7)] Each program helps the students understand mathematics easily and in an enjoyable way.

As we can see from the students' impressions, this model makes the students gain interest and helps them understand the sum of progressions.

If we think based on quantity, 2^2 equals 4^1. However, this model distinguishes 2^2 and 4^1 and identifies 2^2 and 3^2 as being in the same category by surrounding them with rectangles of the same size. Noting the degree of the number, we can find the mathematical construction of the progression, and the formula for the sum of a progression such as $\sum_{k=1}^{n} k^2$ can be easily understood. The "Chips" program helps students understand these concepts. In other words, this model gives the different constructions for these numbers, and it helps students understand these concepts. Professor Miwa wrote that constructive thinking is important since this thinking notes the similarity relationship of the sets rather than the sets themselves and it helps us understand the concepts thoroughly[8)]. Therefore, this model is useful as a mathematical thinking instruction tool.

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