# Modelling with Graphical Representations

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*Traditionally, the modelling of situations in mathematical language consists of creating symbolic expressions, working with and through them using mathematical principles and procedures, and arriving at results which are then reinterpreted in terms of the situation. Technology has provided means to study situations also using graphical models not as mere illustrations but as powerful tools in their own right (to be used before, in parallel, and even instead of symbolic models). Examples of this kind of modelling are presented and their use is illustrated. The educational value of graphical modelling and its implications are discussed.*

## On mathematical models and modelling in school

"… one essential answer (of course not the only one) to the question as to *why* all human beings ought to learn mathematics, is that it provides a means for understanding the world around us, for coping with everyday problems, or for preparing for future professions." (ICMI Study 14, 2002, p. 229). Thus, it is not surprising that applications and modelling is a very important theme in mathematics education. The extent of the interest in this area is evidenced, for example, in the ICMI study now undergoing on the subject, and on the biennial conference held by The International Study Group for Mathematical Modelling and Applications (ICTMA). Besides, comprehensive curricular approaches have been developed and investigated in which situations (from the real world or from within mathematics) are the departure points for learning mathematics. The most salient project with this spirit is the world famous and successful Dutch project called "Realistic Mathematics Education", initiated by the late Professor Hans Freudenthal.

What is modelling? A brief, simple and illuminating definition, which I like, states: "The essence of modelling for me, is a movement between worlds: from the world of the 'problem', … to another familiar world, such as the world of symbols…" Mason (2003, p. 42). Many other descriptions of modelling can be found. For example, "The starting point is normally a certain *situation* in the real world. Simplifying it, structuring it and making it more precise - according to the problem solver's knowledge and interests – leads to the formulation of a *problem* and to a *real model* of the situation. … If appropriate, real data are collected in order to provide more information on the situation at one's disposal … the objects, data, relations, and conditions involved in it are translated into mathematics, resulting in a *mathematical model* of the original situation… mathematical methods come into play, and are used to derive *mathematical results*. These have to be re-translated into the real world, that is *interpreted* in relation to the original situation." (ICMI Study 14, 2002, p. 230).

Some aspects of mathematical modelling are usually taught at school at all levels within any curriculum, for example, word problems in elementary school, distance velocity problems in middle school, and max-min problems in high school calculus (or pre-calculus) courses. Traditionally, (although there are several exceptions) modelling in school mathematics consists of translating a problem into 'mathematical language' (usually symbols), operating with the symbols, and obtaining a solution which is then re-interpreted within the problem situation. Much can be said about the differences between this kind of "mathematical modelling" and the practice of modelling by professionals mathematicians or engineers facing a genuine problem from the real world.

## Modelling with graphs

Rather than discussing how to bridge between the two apparently dissimilar practices of schooling and the professional world, I would like to suggest that much can be done within school mathematics in order to enrich the present practice of mathematical modelling (in most curricula) in order to learn meaningful mathematical content and useful tools in a meaningful way, even when "real" professional practices are not yet fully incorporated.

I would like to refer to mathematical modelling with the following characteristics:

- the usual starting point is a geometrical situation;
- the modelling is being performed with a computerized tool;
- the primary model is a Cartesian graph to express relationships between variables;
- the main purpose is to learn about the situation from its model, and vice versa, to get acquainted with features of graphical modelling on the basis of the situation;
- symbolic models are postponed and their role is re-examined in the light of the above.

In the following, I exemplify these characteristics with several problems, and I discuss them in detail, including the way we use them with students and some of their reactions. The first is a well known problem, which is followed by others less known, which carry some surprises.

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| <u>Problem 1</u>: Given a rectangle of fixed perimeter, 12 cm, investigate the variation of its **area** as a **function** of its **side**. |

A traditional solution path consists of establishing a "mathematical model", namely, translating the conditions of the problem into a symbolic relationship, $A=x(6-x)$ and investigating it. We propose to investigate this problem in a different way, making use of a computerized environment which enables us to model the situation graphically in a dynamic way. In the following (Figure 1), there is a display of a typical screen of such an environment, in which the rectangle of fixed perimeter was constructed, a measurement of its side AB and of its area ABCD was performed, and a Cartesian graph was defined by inputting the dependent variable (in this case the area of the rectangle) into the y-axis and the independent variable (its side) into the x-axis.

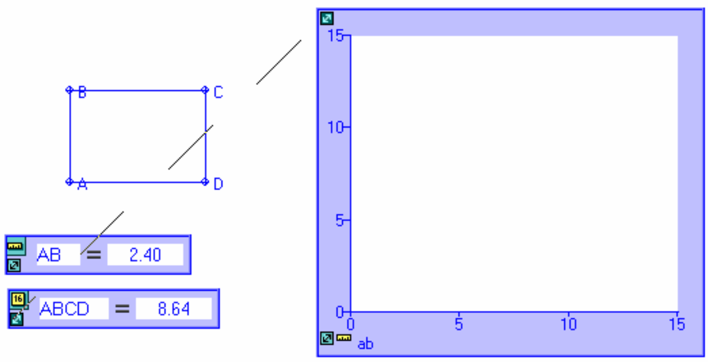


Figure 1: Initial screen displaying the rectangle and the Cartesian graph

Figure 2 shows four screens. On the upper left side of each screen, there are snapshots of the different rectangles as they change dynamically while we drag one of its vertices. The lower left side of each snapshot shows the corresponding measurements (of the area and one side, which change accordingly). Simultaneously, on the right side of each screen, we see the Cartesian graph (of the variation of the area as a function of its side) as it is being traced.

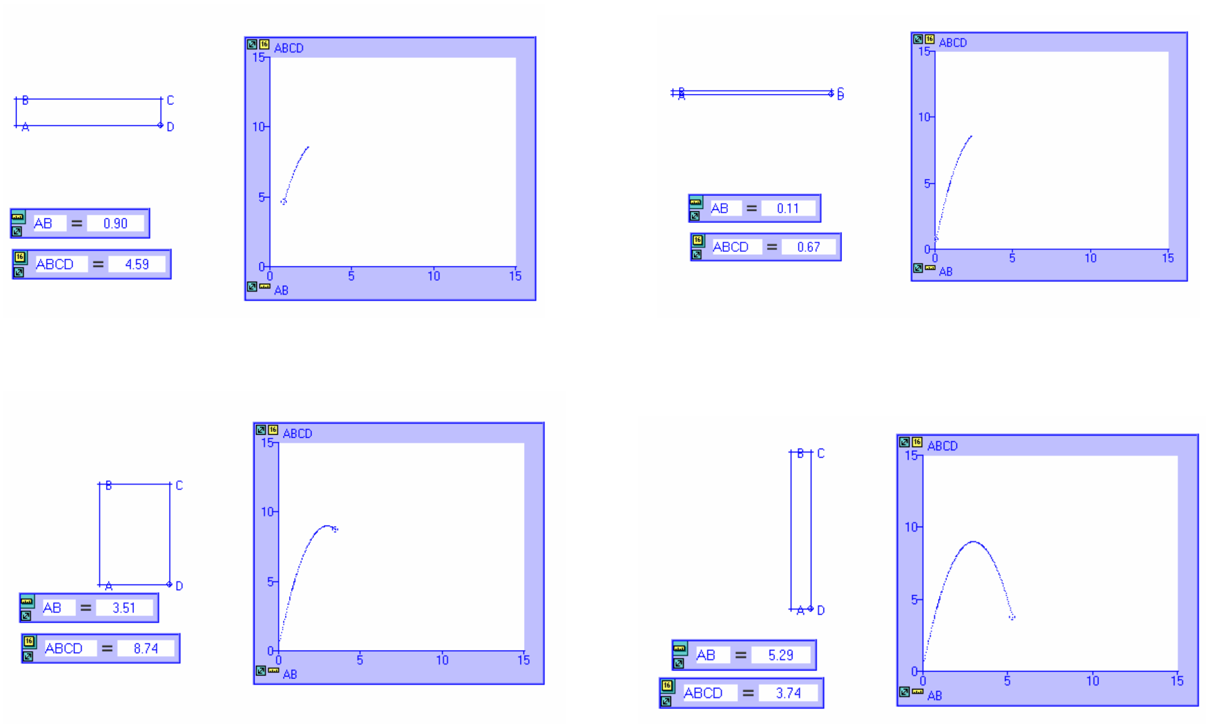


Figure 2: Snapshots of the rectangle changes and the Cartesian graph

This environment enables to study the situation through its graphical model. For example, by considering the following issues:

- When does the maximum occur in the graph and what does it represent?
- What are the graph minima, and what do they represent?
- Explain in terms of the situation the symmetry of the resulting graph.
- When is the area increase slower?

The variation of the area of the rectangle when seen through the graph enables us to visually follow its increase, decrease and rates of change, it highlights the fact that the situation is symmetrical with respect to the “mid-point” of the domain (all the possible values for the length of the side), and it shows that its maximum is attained when the rectangle becomes a square - the only figure that does not “occur” twice when one varies the side over its full domain. In the following, we illustrate further how the graphical model can become an insightful tool for discussing similar issues.

Problem 2: Given a rectangle of fixed perimeter, 12 cm, investigate the variation of its **area** as a **function** of its **diagonal**.

This problem is less common in traditional school textbooks. As previously, we postpone the modelling of this problem with algebraic symbols, and proceed to study first the graphical model.

We propose our students to make a conjecture about the shape of the graph, before proceeding. Figure 3 shows snapshots of the graphing as it is being created.

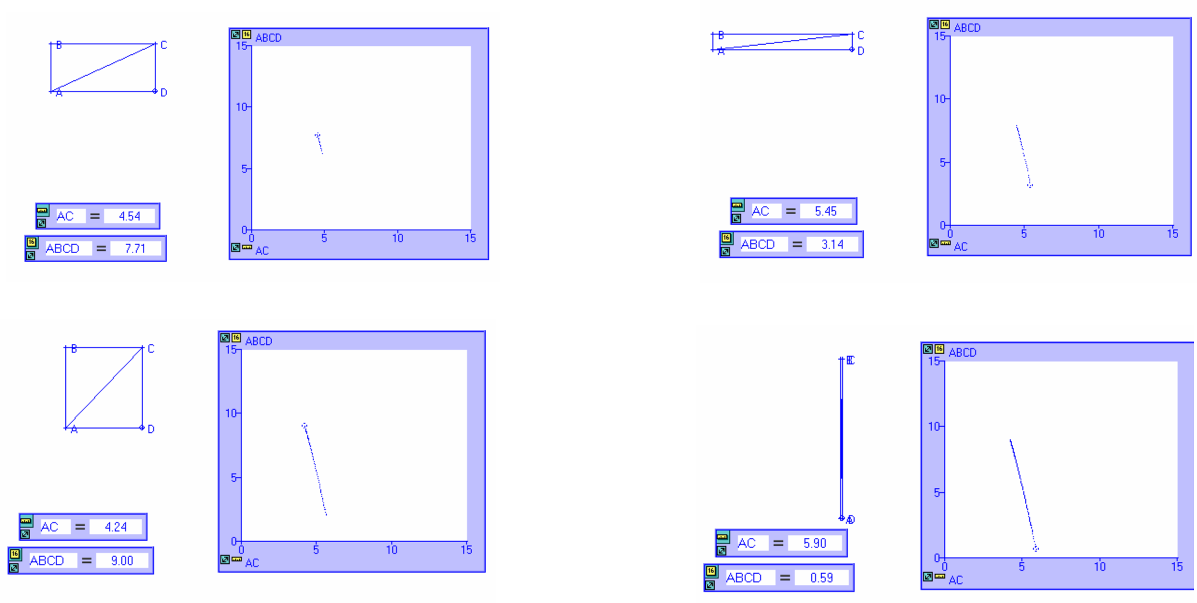


Figure 3: Snapshots of the rectangle changes and the corresponding Cartesian graph

This time the graph is a bit surprising for many. The previous example may have set our minds towards the expectation of a symmetrical graph (parabola), however, here as we change the rectangle dynamically by dragging one of its vertices, the Cartesian graph is being traced as “running” back an forth along what appears to be a segment of a straight line! Is it so? In order to make sense of these surprises, in our trial classrooms, we first investigate the following.

- What are the domain and the co-domain of this function?.
- Explain what are the situations represented by the extreme points of the segment.
- In the previous problem the graph is symmetrical, here it is not. Explain why.
- In Figure 4, both graphs (one from the previous problem and this one) are juxtaposed. Explain a) why the two graphs have the same height, b) the meaning of the proximity of the two graphs, especially when the independent variable increases (do the graphs intersect?, and if so where?)

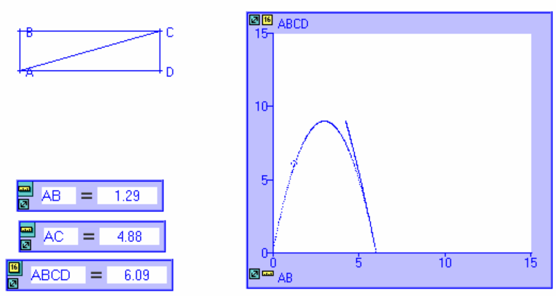


Figure 4: The two graphs juxtaposed

Replying to these questions requires a fruitful interplay between the situation and its graphical model. For example, for many students (and teachers alike) who work with this problem, when looking at the graph they notice for the first time how limited is the domain (all possible values taken by the independent variable) - between a bit more than 4 and less than (about) 6. In order to make sense of this finding, we re-turn our attention to the situation itself and observe with more care the variation of the length of the diagonal as we change the rectangle. We notice that:

a) the diagonal becomes longest when the rectangle “flattens” and it “degenerates” into a segment, namely when the diagonal and the side collapse and both are of length 6, and
b) the shortest diagonal is obtained when the rectangle becomes a square, in which case the diagonal is $3\sqrt{2} \approx 4.24$. Note that the smallest diagonal yields the maximum area.

Another feature of the graphical model which redirects attention to the situation and leads us to re-inspect it, is that the point tracing the graph runs up and down on it as the rectangle changes. This is in contrast to the previous problem, in which the graph was a symmetrical curve. In order to explain this discrepancy between these global graph features, one needs to look again at the situation as it changes dynamically. One realises that the diagonal decreases from one extreme case in which it is longest (when there is no rectangle) to its shortest length (in the case of the square) and then it starts increasing when we “push” the square further. In other words, the independent variable (the length of the diagonal) travels back and forth along its domain for one full cycle of the rectangle changes. This is in contrast to the previous problem, in which the side increases from its smallest value (*0*) to its largest value (6) and on its way “covers” all the rectangles, including those who have the same area (for different values of the domain), thus creating a symmetrical curve.

The juxtaposition of the two graphs highlights features which should be explained. Their equal height is obviously due (but not always immediate for students) to the fact that in both cases the dependent value is the area, thus its maximum value is the same (although is attained for different values of the domain, since the dependent variable is different). The graphs approaching each other reflect the fact that for increasing values of the domain (for the independent variables in both cases) the side and the diagonal approach the same value of *6*.

The educational morale we propose to draw so far is that the dynamical graphical model highlights aspects of the situation which were not as salient had we investigated it alone, or even by modelling it symbolically. In other words, by virtue of the observation and analysis of the graphical model, we notice and explore many features of the situation which become apparent and explicit precisely because they are highlighted by the model.

Let us turn now to a result that puzzled us: is the graph indeed a segment of a straight line? It is useful to stimulate intuitive answers first. For example, some say that it is unlikely that an area as a function of a line segment would be linear, and thus the linearity must be an illusion due to scaling. However, a change of scales to visually discern the type of variation does not help much – in this case the appearance of linearity persists, fueling our doubts further. Certainly, the question cannot be rigorously resolved by graphical means only, and a discussion about the limitations of this model and the need for a different kind of model is in now place. Only the symbolical model will ultimately help us to settle the matter. Thus, in our investigations, we do not rule out the fundamental role of a symbolic model, we just propose to postpone it and to invoke it when we need it, and when it can provide us with precise answers to well formulated questions, which cannot be found elsewhere.

In this case, certain symbolic skill and symbol sense (Arcavi, 1994, 2005) may be needed in order to create a symbolic model. We look for how to express the area as a function of the diagonal, which, for students, may be less straightforward than it seems. The area of a rectangle is $A=xy$, its half perimeter is $x+y$ (in our case is fixed and equal to 6), and its diagonal is $D = \sqrt{x^2 + y^2}$ . This suggests that a good candidate to combine all these expressions in one formula, would be $(x + y)^2 = x^2 + 2xy + y^2$. In our particular case, $36 = x^2 + y^2 + 2A$, yields $A = \frac{36 - D^2}{2}$. This symbolic expression for the function dispels our initial thought that the graph may be linear - the "segment" displayed in the graph must be one of the branches of the parabola, of which only some part of it (the part drawn on the screen) is meaningful as a model for the situation. In order to appreciate how close this branch of the parabola is to a straight line, we ask students to find the equation of the linear function joining the two extremes (namely *(4.24,9)* and *(6,0)*). Once this is obtained, we propose to graph the function of the difference between the linear function and the function of the area. The graph of the difference, in the relevant domain, highlights not only how close these two functions are, but also what is their maximum difference and when it occurs).

The symbolic model can also provide an opportunity to double check if and how all the information we have gathered is represented by the symbols. By inspecting the formula, we can indeed see that when the diagonal is *6*, the area of the rectangle is *0*. However, the symbolic model does not prevent us from substituting any value for *D*, in the interval $0 \le D \le 6$. It is from our acquaintance with the situation, and from the graphical model we explored, that we know that the minimum value for *D* is $3\sqrt{2} \approx 4.24$. If we substitute this value in the area function, we indeed obtain the area of the square of sides 3, which is the maximum possible area (occurring for the minimum value of the diagonal).

At this point, we propose the following.

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| <u>Problem 3</u>: Given a rectangle of fixed perimeter, 12 cm, investigate the variation of its **diagonal** as a **function** of its **side**. |

As with the previous problems, it is very productive to request from students to predict the shape of the expected graph. Making predictions can serve several purposes. Firstly, it engages students with a more careful observation of the situation in order to form an image of the expected result and thus it supports making explicit the students' knowledge and understandings related to the problem. Secondly, prediction creates an affective commitment with the problem and can stimulate classroom discussions. Thirdly, the expectation for a certain result, especially when the actual result is different from the predicted, serves as a strong anchor against which to compare and contrast the expectation with the actual result – such reflection can become a rich source for learning (Arcavi & Hadas, 2000).

From the previous problems, we know that as the side increases from *0* onwards, the diagonal decreases up to a minimum and then starts increasing again, symmetrically. We can also establish some values: when the side is *0*, the diagonal is *6*, it decreases until it reaches the minimum value

$3\sqrt{2} \approx 4.24$, and the it starts increasing until it becomes again *6*. So, most would predict a symmetrical shape, and given the previous experience with problem 1, and the widespread tendency towards prototypicality (as described in, for example, Schwarz & Hershkowitz, 1999) it is very tempting to suggest "parabola". Figure 5 shows the graph drawn.

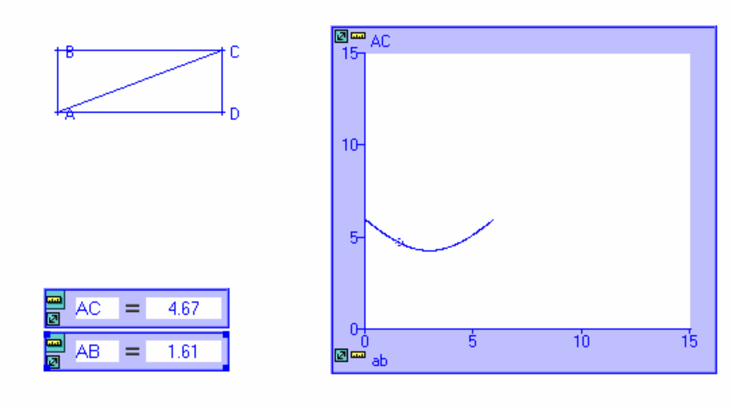


Figure 5: The diagonal as a function of the side

As expected, the graph is symmetrical, with a shape that resembles that of a parabola. However, as we learned from experience, the rigorous resolution can not be done only on the basis of graphs. Indeed, the algebraic model shows that $D = \sqrt{(6-x)^2 + x^2}$, which is not an expression for a parabola. Besides here also, the symbolic form of the function serves as a tool to double check the information gathered from the situation (e.g. extreme values for *x* and corresponding values for *D*, etc).

In the previous problems we explored the variation of the area of a rectangle of fixed perimeter as a function of its side and also as a function of its diagonal. In order to further illustrate the exploration of graphical models, let us now turn to the variation of the area of a fixed rectangle in a way that makes visually perceptible the variation of the area as it occurs. This concretization may well serve as an intuitive visual introduction to the notion of area under a curve (definite integral).

Problem 4: Given a fixed rectangle ABCD, investigate the variation of the **area** of ABEF as a **function** of AF (see Figure 6).

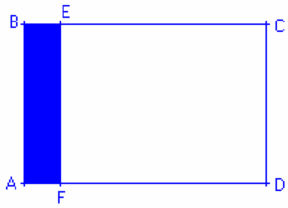


Figure 6: Area of ABEF

Using graphical modelling as our main working tool, we first try to envision the graph of the area of ABEF, as AF changes (increases or decreases). Thus the question posed at this stage is: make a conjecture about the shape of the graph. Some students conjecture that the graph is an horizontal line – due to erroneous visual transfer from features of the situation to features of the graph (this phenomenon was first described in detail by Bell & Janvier (1982)). The following (Figure 7), shows snapshots of the graph as it being created while the side increases dynamically and the area is colored as it increases.

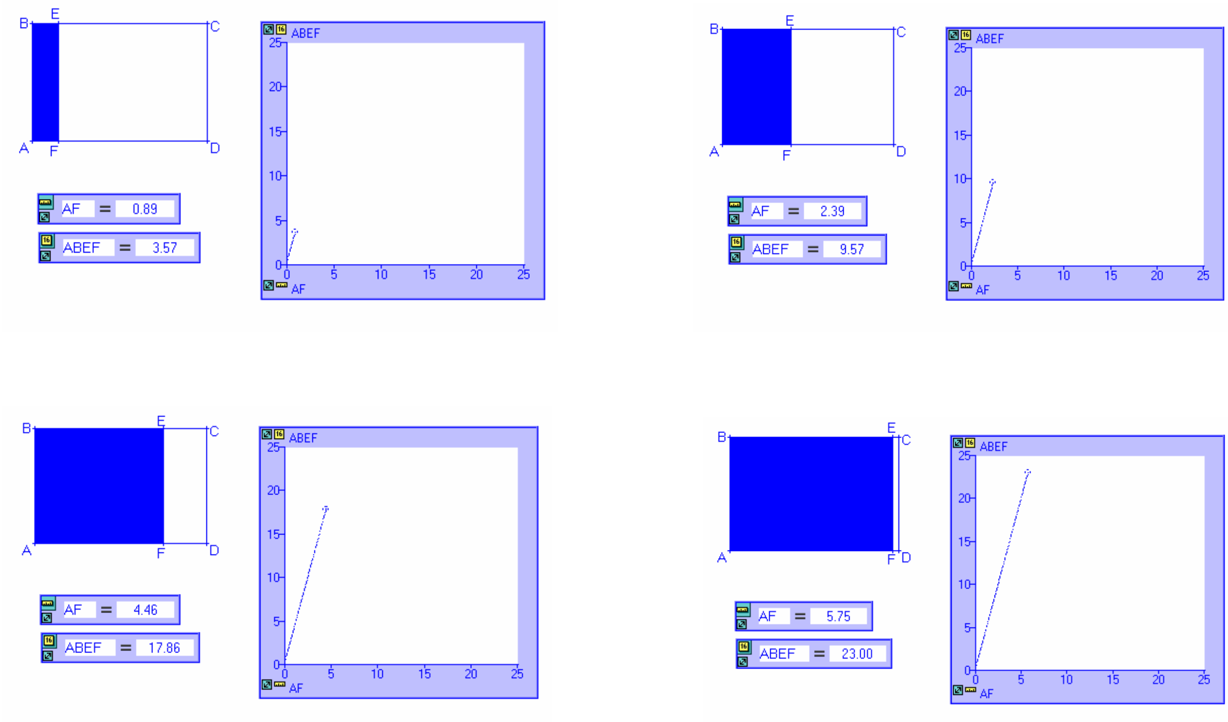


Figure 7: The changing area of a rectangle as a function of one variable side

The graphical model suggests that the change is linear. Is it? In order to elucidate this question, one can proceed in different ways even before resorting to a full-fledged symbolic model (note that the students for whom these problems are intended have no formal knowledge of calculus, and thus it is unlikely that they would connect this problem with the notion of integral as the area under a curve, thus we do not consider this as a possible answer). For example, one may go back to the situation in order to find out whether the characterization of linearity applies there, namely, whether the rate of change is constant. In our case, for the same increases in the length of AF, would we always obtain the same increase in the area of ABEF? Since the area depends only on AF (and also on the width of the rectangle, but the width is constant), the answer is positive – although this type of reasoning is not always immediate for students. A less qualitative way to express this reasoning is to model symbolically the function for the area as *ABEF=a*AF* ,or, if you will, $y=ax$, which describes a linear relationship. In order to make sense of the parameter $a$ in terms of the situation, we ask students how to change the original rectangle ABCD in such a way that the linear graph (obtained when dynamically changing the area of ABEF) will be steeper? In order to answer this question, some students would experiment and obtain a desired graph by trial and error. Others would do an a-priori reflection. Either a-priori or a-posteriori, the reflection will connect among concepts making use of both graphical and symbolic features, for example, as follows. A steeper graph should have a larger slope, namely a greater value of *a* in the formula *y=ax*. Since *a* represents the width of the rectangle, a larger width will yield a steeper graph.

The following problem seems to be a natural follow up inspired by a “what if…?” question. In this case, we propose, what if the rectangle is a triangle?

Problem 5: Given a fixed triangle ABC, investigate the variation of the **area** of ADE as a **function** of AE (see Figure 8).

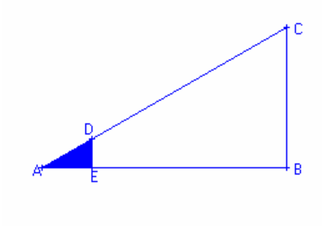


Figure 8: Area of ADE

Again, we propose to pause and make a conjecture about the shape of the graph. Here also, the graphical model may have in store a surprise for many (see Figure 9).

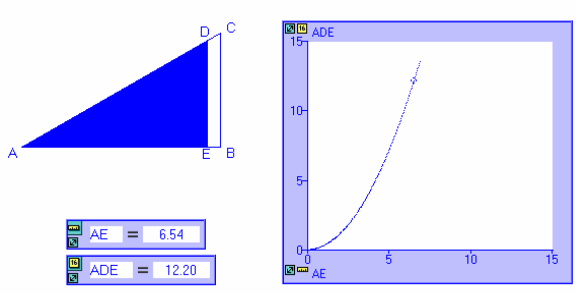


Figure 9: Area of ADE as a function of AE

The graph resembles a parabola, but, is it? how come? A qualitative analysis shows that this case is different from the previous, because, as AE increases, the same increases in the length of AE produce different increases in the area of ADE, thus the rate of change is not constant. Moreover, the graph highlights that the rate of change increases. Can we make sense of this in terms of the situation? The question is aimed at the visual realisation that as we increase AE by equal amounts, the corresponding changes in area increase. Also, we help students to realise that the area of the changing triangle now depends on two variable quantities AE and DE. We notice that the ratio $DE/AE$ is constant, namely that these two variables are proportionally related. Thus, the area of the triangle, $A = (AE * DE)/2$ is a quadratic expression of AE. Now, symbols support what we have found from the graph and the situation in a more formal way.

Let us turn the triangle around and proceed to the next problem.

Problem 6: Given a fixed triangle ABC, investigate the variation of the **area** of ACDE as a **function** of AE (see Figure 10).

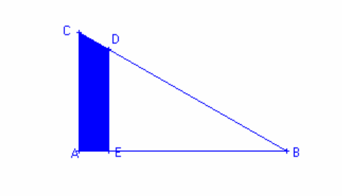


Figure 10: Area of ACDE

What would be the shape of the graph of the area of this trapezoid? Some students claim that the area decreases - this reflects a common confusion stemming from the attribution of the visually salient decrease of the rate of change to the area which surely increases. Once we agree that the area increases as AE increases, we ask whether the graph is linear? As previously, when we increase AE, the same change in AE produce different changes in the area of ACDE. Thus the rate of change cannot be constant, and the function is not linear. Is the graph for this situation similar to the graph of the previous problem? In a sense, it is - the graph corresponds to an increasing function, and it probably is a parabola, however, it must reflect the decrease of the rate of change (See Figure 11).

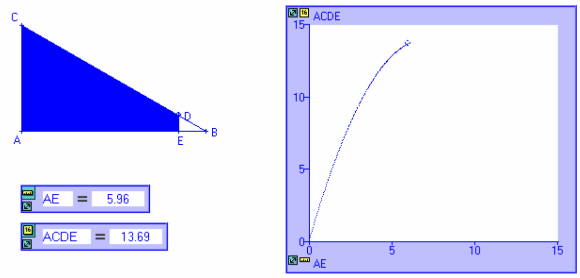


Figure 11: Area of ACDE as a function of AE

Our final problem combines the last problems, as follows.

Problem 7: Explore problems 5 and 6 by simultaneously varying the areas of the triangle and the trapezoid as functions of the same side length (see Figure 12).

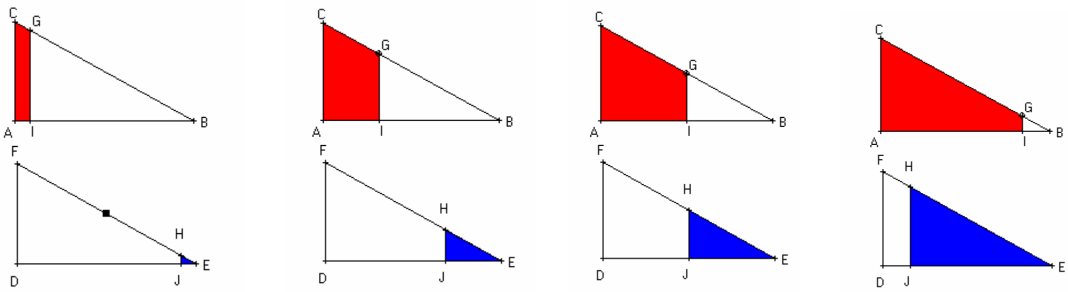


Figure 12: Areas of the trapezoid and the little triangle varying simultaneously

Note that the variation is linked in such a way that the segments AI of the trapezoid and JE of the triangle are always the same. Figure 13 shows the two graphs juxtaposed.

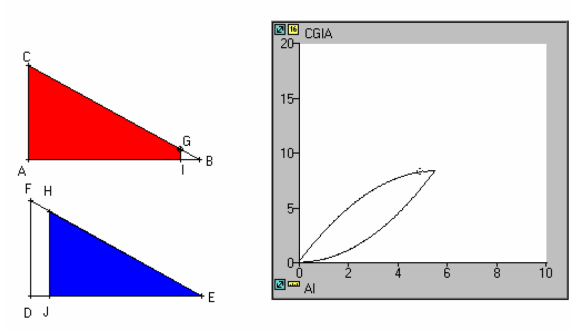


Figure 13: Juxtaposition of the two graphs

The juxtaposition of the graphs suggest some questions, for example, what is the meaning of the intersection points of the graphs? How do you interpret that one of the graphs is fully above the other? Here again, the purpose is to notice interesting features of the graphs and looking back into the situation for what these features represent. We notice, for example, that the area of trapezoid CGIA is always larger than the area of the triangle HJE, and these areas are equal only when we cover all or none of the area of the larger triangles (ABC, EDF).

The next and last issue we propose to students is to conjecture what would be the shape of the graph of the two combined areas (CGIA+HEJ), as a function of the side AI (or JE). This question may seem strange at first, and we implemented in the environment it produces quite a surprise (see Figure 14).

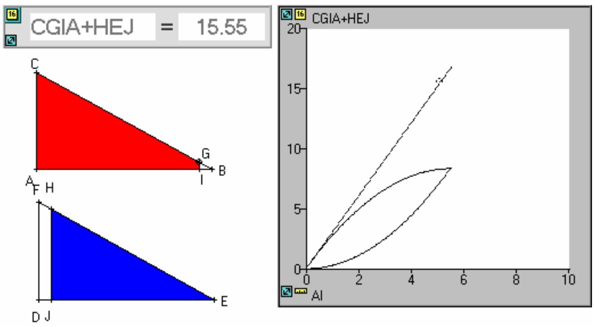


Figure 14: Graph of the two combined areas

The graph of the sum of the areas seems to produce a linear graph! Could it be? How come? The new seemingly linear graph must be the addition, point by point, of the two non-linear graphs (parabolas), is that possible? With these questions in mind we first go back to the situation and then resort to symbols in order to make sense of this puzzling result.

By playing with the figures one starts to visualize what the sum really represents. Figure 15 shows the inversion of one of the two large triangles and its positioning above the other.

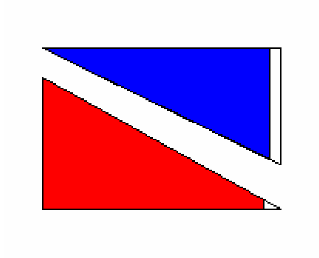

Figure 15: relocation of the two fixed triangles

The sum of the areas add up to a rectangle for which one of its sides is fixed and the other variable - namely we are back in problem 4! Indeed the graph must be a linear. An informal symbolic reasoning will rest us assured that this is perfectly possible: the lower graph is an increasing parabola, the other parabola is decreasing and thus, since the quadratic terms have the same parameter with opposite signs, they cancel out yielding a linear function.

## Final remarks

Modelling geometrical situations with dynamic Cartesian graphs is a promising avenue for new ways of learning mathematical concepts and tools as exemplified above. The following are the main reasons and the possible morals drawn from the work with the above examples.

- Functions
  The concept of function is a central idea in mathematics and it takes a prominent place in most middle and high schools throughout the world. Function is an important mathematical idea in itself, but no less important is its role in developing "students' facility with using patterns and functions to represent, model, and analyze a variety of phenomena and relationships in mathematics problems or in the real world" (NCTM, 2000, p. 227). Also, "teachers may find it helpful to compare and contrast situations that are modeled by functions from various classes" (ibid., p. 297). Function is the core idea of all the activities that model the geometrical situations described above. Moreover, the concept, by virtue of being implemented within a computerized environment, "recovers its dynamism, as a genuine model for change and variation since its graphical representation is being created in real time describing the phenomenon as it occurs" (Arcavi & Hadas, 2000, p. 42)

- Graphical models
  The most widespread representation of functions is the symbolic (algebraic). Traditionally, graphs are invoked later as just an illustrative tool, usually sketched on the bases of features (such as extrema, increase, decrease etc) derived from applying analytic techniques to symbolic expressions. When we model a situation with in this way, some characteristics of the situation may remain opaque, and for many students the activities are technical and mostly devoid of meaning. In the problems above, the graphical model is produced, explored and interpreted *first*. "Both the situation and its graph are looked at dynamically, all information gathered is intimately related and expressed in terms of the situation, and is put at the service of better understanding it." (Arcavi & Hadas, 2000, p. 41). Moreover. whereas a main goal for modelling is to better understand the situation we model, sometimes the situation helps understand the modelling tool better, i.e. certain features of the graphs (as "addition of graphs" in the last problem above).

- Symbolical models
  In the above examples, symbols are introduced when we need them in order to transcend limitations of the graph or when we want to re-inspect in a different way information already gathered. In this way, rather than engaging in mostly technical manipulation of symbols, the “algebraic expression comes alive…” (Noss & Hoyles, 1996, p.245) - it is preceded by meaning making activities that drive the way we look at symbols, use them and interpret them. This trajectory (graphs first, symbols later) by no means imply to use each of the models as a watertight compartment. Quite the contrary, once all the models are introduced, the interplay among them is very fruitful, as illustrated in most of the examples.

- Surprises
  Surprises can be a good source for learning: the contrast between an expected outcome and the actual surprising outcome is the space in which meaningful and productive questions are asked by students themselves, followed by attempts at resolving them. However, it is up to us, mathematics educators, to set the scene for these surprises to occur. A first challenge is to design activities in which hands-on (and “minds-on”) experimentation leads to puzzling and unexpected results. A second challenge is to nudge students to explicate their predictions in order to make visible the sources of their reasoning and conceptualizations and also as a means to make them more committed to the task. Explicit predictions, or conjectures, make the surprise (if/when it happens) more meaningful and provides rich opportunities to revise our knowledge, its sources and the tools we use. A third challenge consists of conducting classroom discussions, which take maximum advantage of the discrepancies between the predictions and the actual results.

- Experimental arena
  The computerized environment we use constitutes a rich experimental arena. Students receive the feedback as a direct consequence of their actions and not as a judgmental statement from their teachers. Besides, students can use the environment as a springboard for conversation and discussions with their peers and teachers, are free to experiment, to produce questions of their own, and to embark in further explorations.

- Intuitive introduction to formal topics
  All the characteristics above combined offer us new learning paths and opportunities, mostly by helping to build informal intuitive knowledge infrastructures as sound foundations for the more advanced mathematics. It is our belief that, such foundations would prevent many of the student failures in higher mathematics, which, in my opinion, are mostly due to the lack of sense making we hereby propose to stimulate and support.

## Coda

We started with a shortened quote from Mason (2003) about modelling. We conclude with the quote in full, because, as we understand it, it faithfully reflects the spirit of the present work. “The essence of modelling for me, is a movement between worlds: from the world of the ‘problem’, through the world of imagery in which an ‘essence’ is sought in the abstract, the pure, the ideal, the simplified, to another familiar world, such as the world of symbols, of scaled down material objects, or of pre-made simulations. Finally there is a movement back through imagery again to the original problematic situation, and this cycling may be repeated several times at various levels of detail before some sort of conclusion is reached and recorded” (ibid., p.42).

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