Java Geometry Expert
and its Applications to Geometry Education

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Abstract: The Java Geometry Expert (Java GEX) is a piece of software for dynamic geometry. Java GEX also has the powerful proof methods including Wu's method based on algebraic computation, the area method based on geometric invariants, and the deductive database methods based on the theory of logic programming. Since Java is platform-independent, people can use Java GEX at any platform.

1. Introduction

Since the pioneering work by the eminent Chinese mathematician Wen-Tsun Wu [6], great achievements have been made by world-wide researchers in geometry theorem proving. Hundreds of difficult theorems whose traditional proofs require enormous amounts of human intelligence, such as Feuerbach's theorem, the Morley trisector theorem, etc., have been proved totally automatically by computer programs based on Wu's algorithm. Inspired by the success of Wu's method, many other approaches to automated proving of geometry theorems have been proposed. For a recent survey, please consult [2].

The software Geometry Expert (GEX) was originally developed around 1994 [3]. It consists of two parts: the proving and reasoning part, and the drawing part. The dynamic nature of its drawing part is comparable to that of Cabri and the Geometer's Sketchpad. As a geometry theorem prover, GEX implements Wu's method as well as the area method [4] and the deductive basis method [2] which can generate elegant and short (sometimes even shorter than those given by geometry experts) proofs.

However, GEX uses Openwin under X-Window which is no longer supported by the Linux distributions after 2000. Thus we could not satisfy the requests for GEX from students and researchers after 2000.

2. Java Geometry Expert

The Java Geometry Expert (Java GEX) has been rewritten completely with emphasis on its ease of use by high school students and teachers in geometric drawing for educational purposes. Java GEX will still have the powerful proof methods of GEX including Wu's method based on algebraic computation, the area method based on geometric invariants, and the deductive database methods based on the theory of logic programming.

The primary goal of Java GEX is to provide a piece of alternative and free software for dynamic geometry which can also prove theorems automatically. We began this project in 2004 and plan to release the first official version in 2006.

Since Java is platform-independent, people can use Java GEX at any platform by downloading the compiled Java bytecode to their local machines. Also any user with a browser and the INTERNET connection can use a major part of our Java GEX on our Web Server:

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to see examples shown in Section 3. We have prepared a few dozens of examples. Please click the example menu item to play with these examples. In these examples, points with the red or green color are free or semi-free points that can be dragged with the mouse. The user can see the construction of a diagram step by step, or can animate the diagram in deferent settings.

2.1. Geometric and Non-Geometric Drawing

Currently, our main effort concentrates on the graphics drawing and the dynamic geometric diagrams.

There have been excellent pieces of software for geometric drawing, e.g., Cabri, the Geometer's Sketchpad, and Cinderella. However, they lack the reasoning and proving abilities. Hence their drawing is non-geometric in the sense that when a geometric diagram with a particular property is drawn, they can only numerically verify this property. We call this kind of drawings *non-geometric*. Non-geometric drawings have many advantages. For example, if we want to draw a regular n-polygon, the computer can easily calculate the degree of an angle of this regular polygon. Also it can use the cut/paste technique for a portion of the diagram just as the Geometer's Sketchpad does. We plan to incorporate these non-geometric drawing techniques in the late stage of developing our system.

At right beginning of developing the drawing part of Java GEX, we have kept the proving and reasoning as our final goal since we treat geometric elements, such as points, lines, and circles symbolically. Then we can apply the reasoning methods developed for over a quarter of a century to proving and/or discovering properties of a geometric diagram constructed. For example, if we want an angle of degree 60, we need to construct an equilateral triangle (see Example 3.6: the three parallel line construction problem).

We call this kind of drawing *geometric*. If a part of a diagram is geometrically constructed, we can reason on this part. An example is shown in the next section.

2.2. An Application of Geometric Drawing

If a new point constructed is identical to one of the previous constructed points, non-geometric drawings can only eliminate this point numerically. Since drawing a geometry diagram is so easy with few mouse clicks, we repeatedly observe this phenomenon in the early stage of developing Java GEX.

Take an isosceles triangle ABC with the base segment BC as an example. If reflecting AB wrpt BC, we get a segment BA'; then reflecting AC wrpt BC, we get another segment CA". Points A' and A" are actually identical.

This is a simple example, but we have also encountered cases in which the identity itself is a relatively deep theorem. Our approach to removing the identical point is as following.

Whenever a new point is constructed, we first check whether this point is identical to a previous one numerically. If so, we use Wu's method to check the identity. If the identity is valid, we eliminate this point.

3. Examples

Here we use our preliminary version of Java GEX to show how to automatically prove theorems of equality type, how to manually construct a diagram with the ruler and compass, how to draw diagrams of inequality theorems for manual proofs.

3.1. Theorems of Equality Type

In terms of geometry, geometry theorems of equality type are those that do not involve the order relation e.g., point C is between points A and B. There are five groups of axioms for elementary geometry in Hilbert's classic book [5]. The second group is axioms of order. However, further developments have shown that there are unordered geometries that do not use axioms of order. Wu's method and the area method are amenable to these geometries. For detailed discussions, see [7,1]. For example, the traditional proof of the following example in geometry books uses the order relation. However, Wu's method or the area can prove it without using the order relation.

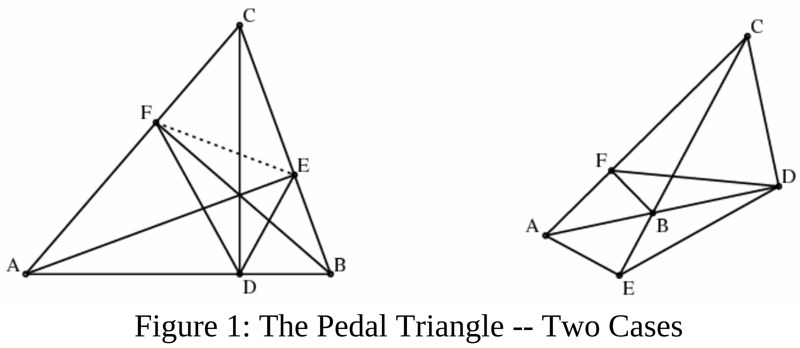

Figure 1: The Pedal Triangle -- Two Cases

Java GEX will be able to prove these theorems as it has done in the books and papers [1,4,3].

Example 3.1 (The Pedal Triangle) As shown in Figure 1, let D, E, and F be the feet of the three altitudes of triangle ABC. Prove that ∠EDC = ∠CDF.

Note that there are two cases when ∠ABC < 90° and ∠ABC > 90°. If ∠ABC > 90°, ∠EDC and ∠CDF are complement and have the opposite orientations. Wu's method proves these two cases to be valid since here we use full-angles instead of traditional angles [4]. When triangle ABC is acute, we call triangle DEF the pedal triangle of triangle ABC. Thus the altitudes of an acute triangle are the (internal) bisectors of its pedal triangle. We will use this fact when we discuss the Schwarz solution to the pedal triangle problem: Example 3.5.

Example 3.2 (The Nine Point Circle Theorem) In a triangle ABC, the following nine points are on the same circle: the feet of the three altitudes, the midpoints of the three sides, and the midpoints from three vertexes to the orthocenter, point G in the two diagrams in Figure [2].

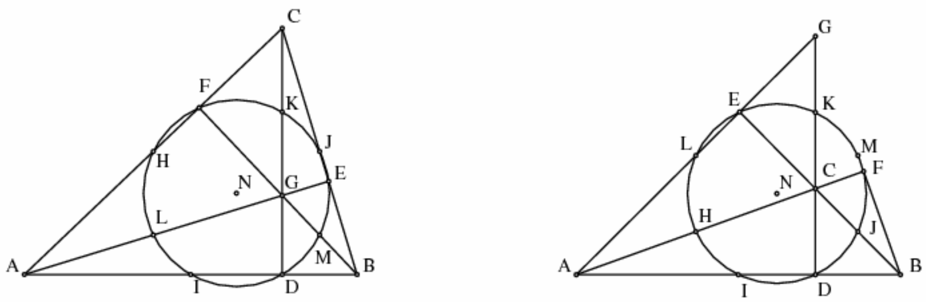

Figure 2: Nine Point Circle: two cases.

Note that there are two cases when ∠ ACB < 90° and ∠ACB >= 90°. These two cases can be seen from the animation when vertex C is moving along the altitude CD. Hence triangles ABC and triangle AGB are symmetrical in the Nine Point Circle Theorem.

Example 3.3 (Feuerbach's Theorem) The Nine Point Circle of a triangle is tangent to the inscribed circle as well as the three excribed circles of the triangle.

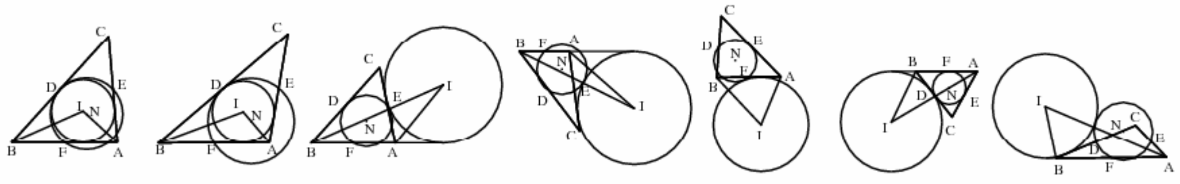

Figure 3: Feuerbach's Theorem

Figure 3 contains some cases in animation showing that Wu's method proves that the nine point circle is tangent to all 4 circles (one inscribed and 3 excribed); for detail see [1].

3.2. Theorems Proved by Human

Example 3.4 (The Pythagoras Theorem) $a^2 + b^2 = c^2$.

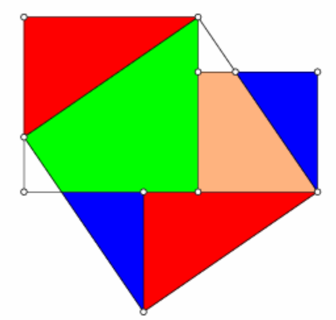

Figure 4: A Human Proof of the Pythagoras Theorem

There are a few dozen proofs of this theorem, a quite few of which use the area cut/paste method. The following is one of our favorites. As shown in Figure 4, the three pairs of triangles with the same color (white, red, and blue) are congruent; the two quadrilaterals (green and light brawn) are common among a^2 and c^2 and, among b^2 and c^2; hence the proof is shown from the diagram. If the hard copy of this pdf file is non-colored, please see our Web Server in which the construction of Figure 4 is displayed step by step. At the steps that two congruent triangles with the same color appear, the colored triangles are blinking a few times. Thus the proof is much clearly visualized.

Example 3.5 (The Pedal Triangle Problem) Find three points on the three sides of an acute triangle so that the perimeter of the triangle formed by these three points is minimal.

Although the answer was known when this problem was proposed, but H. A. Schwarz gave an elementary and elegant solution to this problem by reflecting the triangle five times. Then the straight segment FU is twice of the perimeter of the pedal triangle.

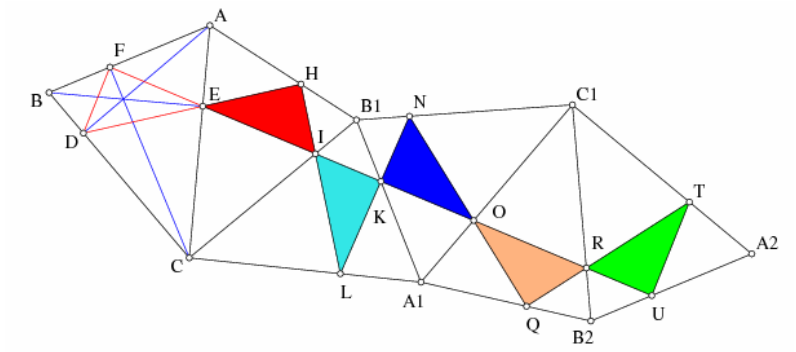

Figure 5: The Schwarz solution to the pedal triangle problem

3.3. Construction Problems with the Ruler and Compass

We proposed a method for solving ruler and compass construction problems [2]. About 130 problems have been solved by this method. The program is in Prolog because the depth-search and the breadth-search strategies can be easily implemented in Prolog.

However, this method is incomplete; many problems that can be constructed with the ruler and compass cannot be solved by our program. We plan to extend this method further and implement it in Java GEX with elegant graphics interface. The following problem cannot be solved by our method developed so far. However, once we find a solution manually, we can easily construct the diagram with Java GEX using the ruler and compass.

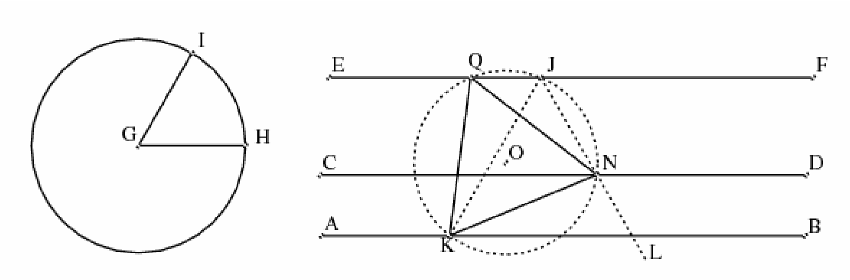

Figure 6: The Three Parallel Line Problem

Example 3.6 (The Three Parallel Line Problem) Construct an equilateral triangle with its three vertexes on three given parallel lines.

Here is a ruler and compass solution. First we need an angle with 60°, i.e., construct an equilateral triangle IGH. Pick any point J on line EF and construct the angle EJK to be equal to $\angle$IGH (60°) with K being on line AB. Then reflect line EF wrpt line JK to obtain line JL. Let N be the intersection of line CD and JL. Draw the circumscribed circle of triangle JNK. Let Q be the other intersection of the circumscribed circle and line EF. Then triangle QKN is an equilateral triangle.

Since our construction is geometric, Wu's method can prove triangle QKN to be equilateral.

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