The Center of Gravity of Plane Regions and Ruler and Compass Constructions

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Abstract: In this paper we will show how to use a computer algebra system (CAS) to study various aspects of the center of gravity of plane regions. The study includes trapezoidal, parabolic, and exponential regions. We first consider such regions with fixed boundaries. In order to calculate their center of gravity, one needs to calculate certain integrals, which could be tedious. A CAS such as *Mathematica* is a useful tool for such calculations. Next we will consider such regions with variable boundaries. These boundaries change with a certain parameter. As the parameter changes, the center of the gravity of the region changes, and *Mathematica* can be used to study the locus of the center of gravity. We will also show how to use *Mathematica* to make animations of the center of gravity as the parameter changes. Some theorems on the center of gravity of these regions will be obtained. The final section is devoted to the ruler and compass constructions of the center of gravity of some of those regions.

1. Introduction

Consider a closed plane region $\mathcal{R}$ in the *XY*-plane. Recall that the center of gravity $G(\bar{x}, \bar{y})$ of the region $\mathcal{R}$ is given by (see [10], [11], [12], and [13])

$$\bar{x} = I_x / I \tag{1.1}$$

$$\bar{y} = I_y / I \tag{1.2}$$

where the integrals I_x, I_y, and I are defined by the double integrals

$$I_x = \iint_{\mathcal{R}} x \, dA \tag{1.3}$$

$$I_y = \iint_{\mathcal{R}} y \, dA \tag{1.4}$$

and

$$I = \iint_{\mathcal{R}} dA \tag{1.5}$$

One can now consider the following special case: Suppose a and b are real constants such that $a < b$. Suppose that f and g are continuous functions on the closed interval $[a,b]$ such that $f(x) \geq g(x)$ for all x in $[a,b]$. Let $\mathcal{R}$ be the region bounded by the graphs of

$y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. Then the equations (1.1)-(1.5) imply that the center of gravity $G(\bar{x}, \bar{y})$ of the region is given by the following equations:

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b (f(x) - g(x))dx} \qquad (1.6)$$

and

$$\bar{y} = \frac{\int_a^b (1/2)(f(x) + g(x))(f(x) - g(x))\,dx}{\int_a^b (f(x) - g(x))dx} \qquad (1.7)$$

It is useful to realize that the denominator of the above equations (1.6) and (1.7) represents the area of the region $\mathcal{R}$. In the next few sections, we will use the equations (1.6) and (1.7) to calculate the center of gravity of several types of regions. Out first region has the shape of a trapezoid.

2. The Center of Gravity of a Trapezoidal Region

Suppose that a and b are real constants such that $a < b$, and f is the linear function defined by $f(x) = px + q$ for x in $[a,b]$ where p and q are constants. Let us assume that $q > 0$, and p is such that the graph of f intersects the vertical lines $x = a$ and $x = b$ on the upper-half plane. Let $\mathcal{R}$ be the trapezoidal region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the x-axis. See the following figure:

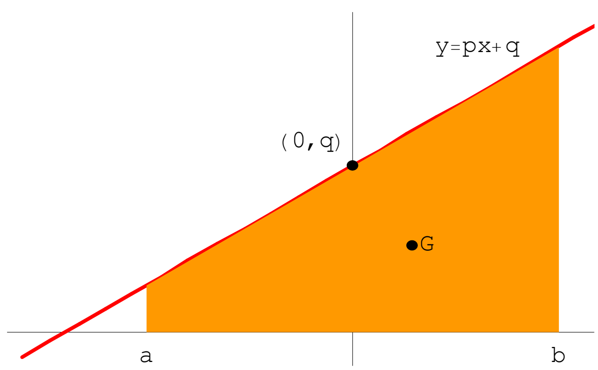

Figure 2.1 The Center of Gravity of a Trapezoid

One can use the equations (1.6) and (1.7) with $g(x) = 0$ to calculate the center of gravity $G(\bar{x}, \bar{y})$ of the region $\mathcal{R}$. For example, the denominators of the equation (1.6) and (1.7) can be calculated as

$$I = \int_a^b (px + q)dx = p[x^2/2]_a^b + q[x]_a^b = (p/2)(b^2 - a^2) + q(b - a)$$

The numerator of equation (1.6) reads as

$$I_x = \int_a^b x(px+q)dx = p[x^3/3]_a^b + q[x^2/2]_a^b = (p/3)(b^3 - a^3) + (q/2)(b^2 - a^2)$$

However, calculating the numerator of the equation (1.7) is a bit more tedious:

$$I_y = \int_a^b (1/2)(px+q)^2 dx = (p^2/2)[x^3/3]_a^b + (pq)[x^2/2]_a^b + (q^2/2)[x]_a^b$$
$$= (p^2/6)(b^3 - a^3) + (pq/2)(b^2 - a^2) + (q^2/2)(b-a)$$

After some simplification, the above three equations together with the equations (1.6) and (1.7) imply that the center of gravity $G(\bar{x}, \bar{y})$ of the region $\mathcal{R}$ is given by

$$\bar{x} = \frac{2p(a^2+ab+b^2)+3q(a+b)}{3(ap+bp+2q)} \tag{2.1}$$

$$\bar{y} = \frac{p^2(a^2+ab+b^2)+3pq(a+b)+3q^2}{3(ap+bp+2q)} \tag{2.2}$$

One can use a CAS such as *Mathematica* to facilitate the above calculations. For example, the **"Integrate"** command of *Mathematica* can be used to calculate the integrals involved in formulas (1.6) and (1.7) (see [9] and [15]). The following *Mathematica* program automates the task of finding the center of gravity of our trapezoidal region $\mathcal{R}$:

Program 2.1

```
f[x_]:=p*x+q
g[x_]:=0
ix=Integrate[x*(f[x]-g[x]),{x,a,b}];
iy=Integrate[(1/2)(f[x]+g[x])(f[x]-g[x]),{x,a,b}];
i=Integrate[f[x]-g[x],{x,a,b}];
{xbar,ybar}=Simplify[{ix/i,iy/i}]
```

Press **"Shift-Enter"** to execute the program. As the output we get the following coordinates of the center of gravity:

$$\left(\frac{2a^2p+2abp+2b^2p+3aq+3bq}{3ap+3bp+6q}, \frac{a^2p^2+abp^2+b^2p^2+3apq+3bpq+3q^2}{3ap+3bp+6q}\right)$$

One can readily see that the above output agrees with equations (2.1) and (2.2) giving the coordinates of the center of gravity *G*. The program 2.1 can be used to experiment with other types of regions as we shall soon see. In fact, *Mathematica* can be used more than just a tool for computation in our studies of center of gravity (see section 4 of this paper).

Mathematica is a general purpose CAS. It can be used as a tool for numeric or symbolic computation, a tool for two or three-dimensional graphing, a visualization device, a programming language, or even as a multimedia studio combining sounds and graphics. Some general references on *Mathematica* are [1], [9], [14], and [15]. For the usage of *Mathematica* as a visualization tool refer to [3], [4], [6], and [7]. For the usage of *Mathematica* as a pattern recognition and conjecture-forming tool, refer to [2], [3], [5], and [8].

In the next section, we will study the center of gravity of parabolic and exponential regions.

3. The Center of Gravity of Parabolic Regions and Exponential Regions

(a) *Parabolic Regions*

Suppose that a, b, p, and q are real constants with $a > 0$ and $b < q$. Let S be region bounded by the graphs of $f(x) = px + q$ and $g(x) = ax^2 + b$. Let $G(\bar{x}, \bar{y})$ denote the center of gravity of the region S. See the figure below.

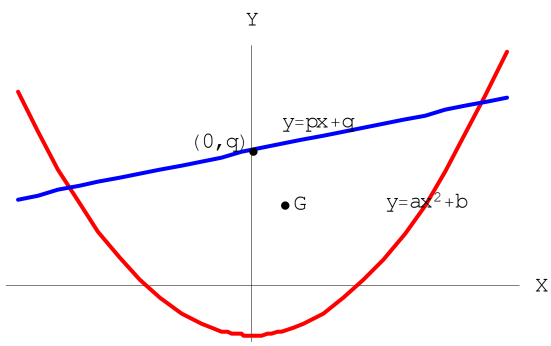

Figure 3.1 The Center of Gravity of a Parabolic Region

Let c and d be the x-coordinates of the points of intersection of the two graphs $y = f(x)$ and $y = g(x)$ with $c < d$. Then the equations (1.6) and (1.7) imply that

$$\bar{x} = \frac{\int_c^d x[(px+q)-(ax^2+b)]dx}{\int_c^d [(px+q)-(ax^2+b)]dx} \tag{3.1}$$

$$\bar{y} = \frac{\int_c^d (1/2)[(px+q)^2-(ax^2+b)^2]dx}{\int_c^d [(px+q)-(ax^2+b)]dx} \tag{3.2}$$

The integrals in equations (3.1) and (3.2) are a lot harder to compute by hand compared to the corresponding calculations in the previous section. *Mathematica* will greatly facilitate our computations. For example, to find the limits of integration c and d, we can use the **"Solve"** command of *Mathematica* (see [9] and [15]). The following program calculating the center of gravity of the parabolic region $\mathcal{S}$ was written by modifying the program 2.1 in the previous section.

Program 3.1

```
f[x_]:=p*x+q;
g[x_]:=a*x^2+b;
{c,d}=x/.Simplify[Solve[f[x]==g[x],x] ];
ix=Integrate[x*(f[x]-g[x]),{x,c,d}];
iy=Integrate[(1/2)(f[x]+g[x])(f[x]-g[x]),{x,c,d}];
i=Integrate[f[x]-g[x],{x,c,d}];
{xbar,ybar}=Simplify[{ix/i,iy/i}]
```

According to the output of the program, the center of gravity G of the parabolic region $\mathcal{S}$ is given by

$$\bar{x} = p/(2a) \tag{3.3}$$

$$\bar{y} = (2ab + 2p^2 + 3aq)/(5a) \tag{3.4}$$

(b) *Exponential Regions*

Let a and b be real constants such that $a < b$. Consider the exponential function $f(x) = pe^{kx} + q$ where p, q, and k are real constants with $p, q > 0$. Let $\mathcal{T}$ be the region bounded by the graphs of $y = f(x)$, $x = a$, $x = b$, and the x-axis. As usual, let $G(\bar{x}, \bar{y})$ denote the center of gravity of the exponential region $\mathcal{T}$. See the following figure:

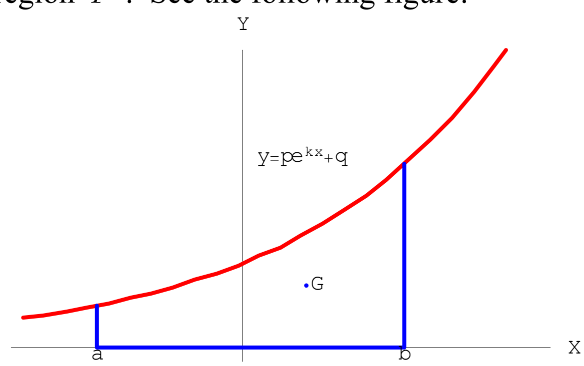

Figure 3.2 The Center of Gravity of an Exponential Region

Like before, one can write the following *Mathematica* program to calculate the center of gravity of the region $\mathcal{T}$:

Program 3.2

```
 f[x_]:=p*Exp[k*x]+q
g[x_]:=0
ix=Integrate[x*(f[x]-g[x]),{x,a,b}];
```

```
iy=Integrate[(1/2)(f[x]+g[x])(f[x]-g[x]),{x,a,b}];
i=Integrate[f[x]-g[x],{x,a,b}];
{xbar,ybar}=Simplify[{ix/i,iy/i}]
```

According to the output of the program, the center of gravity G of the exponential region $\mathcal{T}$ is given by

$$\bar{x} = \frac{2pk(ae^{ak} - be^{bk}) + 2p(e^{bk} - e^{ak}) + (a^2 - b^2)k^2 q}{2k[p(e^{ak} - e^{bk}) + (a-b)kq]} \quad (3.5)$$

$$\bar{y} = \frac{p^2(e^{2ak} - e^{2bk}) + 4pq(e^{ak} - e^{bk}) + 2(a-b)kq^2}{4[p(e^{ak} - e^{bk}) + (a-b)kq]} \quad (3.6)$$

The next section adds an interesting twist to the center of gravity problems we have so far presented. Consider any of the trapezoidal, parabolic, or exponential regions described previously. What will happen to the center of gravity if we allow one of the boundaries of the region to change gradually? In this way, we can study the movement of the center of gravity in the *XY*-plane. Thus we have an entire collection of locus problems at our disposal.

4. The Locus of the Center of Gravity of Variable Plane Regions

(a) *Variable Trapezoidal Regions*

Let us again consider the trapezoidal region $\mathcal{R}$ discussed in section 2 of the paper. The four boundaries of this region were $x = a, x = b, y = 0,$ and $y = px + q$. Suppose now we fix the constants *a*, *b*, *q*, and allow the slope *p* of the top boundary to vary. As a result, the top boundary of the region will tilt around its fixed y-intercept $(0, q)$. Therefore, the region $\mathcal{R}$ changes, and we are interested in studying the locus (path) of its center of gravity $G(\bar{x}, \bar{y})$ in the *XY*-plane.

In order to find the equation of the locus of *G*, one must eliminate the parameter *p* between the equations (2.1) and (2.2). For example, the following **"Eliminate"** command of *Mathematica* will perform the required task (see [9] and [15]):

Input:

```
Eliminate[{x, y} == {(2a^2*p + 2a*b*p + 2b^2*p + 3a*q + 3b*q)/ (3a*p + 3b*p + 6q),
    (a^2*p^2 + a*b*p^2 + b^2*p^2 + 3a*p*q + 3b*p*q + 3q^2)/ (3a*p + 3b*p + 6q)}, p]
```

Output: $3qx^2 + x(-3aq - 3bq + 3ay + 3by) == -a^2q - abq - b^2q + 2a^2y + 2aby + 2b^2y$

Use the **"Solve"** command of *Mathematica* to solve the above equation for *y*:

Input:

```
Solve[3q*x^2 + x (- 3a*q - 3b*q + 3a*y + 3b*y) = = -a^2*q - a*b*q -b^2*q
    + 2a^2*y + 2a*b*y + 2b^2*y, y]
```

Output: $$\left\{\left\{y \to \frac{-a^2 q - abq - b^2 q + 3aqx + 3bqx - 3qx^2}{-2a^2 - 2ab - 2b^2 + 3ax + 3bx}\right\}\right\}$$

The output means that the locus of the center of the gravity of the variable trapezoidal region $\mathcal{R}$ is given by

$$y = \frac{-q(a^2 + ab + b^2) + 3qx(a+b) - 3qx^2}{3x(a+b) - 2(a^2 + ab + b^2)} \tag{4.1}$$

The above equation represents a rational function in *x*. As a special case, if $a + b = 0$, the above rational function reduces to the following quadratic function:

$$y = \frac{3q}{2a^2}x^2 + \frac{q}{2} \tag{4.2}$$

The above equation (4.2) means that if $a + b = 0$, the locus of the center of gravity of the corresponding trapezoidal region is a parabola opening up with *y*-intercept $q/2$. We can summarize our findings in the following theorem.

Theorem 4.1 Suppose that *a* and *b* are fixed real constants such that $a < b$, and let f be the linear function defined by $f(x) = px + q$ for x in $[a,b]$ where *p* is a real parameter and *q* is a fixed real constant. Let us assume that $q > 0$, and *p* is such that the graph of *f* intersects the vertical lines $x = a$ and $x = b$ on the upper-half plane. Let $\mathcal{R}$ be the trapezoidal region bounded by the graphs of $y = f(x)$, $x = a, x = b,$ and the *x*-axis. Then for changing *p*, the equation of the locus of the center of gravity of the region $\mathcal{R}$ is a rational function given by equation (4.1). For the special case $a + b = 0$, this locus reduces to a parabola given by equation (4.2).

Proof. Can be proved independent of *Mathematica* calculations.

So far in this paper, we used *Mathematica* as a computational tool. One can also use *Mathematica* to visualize the movement of the center of gravity for changing parameter *p*. For example, the following program creates an animation of the center of gravity of the variable region $\mathcal{R}$.

Program 4.1

```
Clear[a,b,p,q]
f[x_]:=p*x+q
g[x_]:=0
i1=Integrate[x(f[x]-g[x]),{x,a,b}];
i2=Integrate[(1/2)(f[x]-g[x])(f[x]+g[x]),{x,a,b}];
i=Integrate[f[x]-g[x],{x,a,b}];
{xbar,ybar}=Simplify[{i1/i,i2/i}];
expr=y/.Solve[Eliminate[{x,y}=={xbar,ybar},p],y][[1]]
```

```
a=-1;
b=2;
q=2;
Do[Plot[{f[x],expr},{x,a,b},PlotRange->{0,5.5},
        PlotStyle->{{Thickness[1/80],RGBColor[1,0,0]},{RGBColor[0,1,0]}},
        Prolog->{ {RGBColor[1,0.6,0],Polygon[{{a,0},{a,f[a]},{b,f[b]},{b,0}}]},
            {RGBColor[0,0,1],Thickness[1/100],Line[{{a,0},{a,f[a]}}]},
            {RGBColor[0,0,1],Thickness[1/100],Line[{{b,0},{b,f[b]}}]},
            {RGBColor[0,0,1],Thickness[1/100],Line[{{a,0},{b,0}}]},
            {RGBColor[1,0,0],PointSize[1/60],Point[{xbar,ybar}]} }],
                    {p,-0.8,1.5,0.05}]
```

The above program achieves more than one task: First of all it calculates the coordinates of the center of gravity *G*. Then by eliminating the parameter *p* between two equations, it calculates the equation of the locus of *G*. Then for given specific values of *a*, *b*, and *q*, the program creates an animation of the point *G*. When the animation is run, one can notice that the upper boundary of the trapezoidal region is tilting around its fixed *y*-intercept. As the region is changing, observe that the point *G* (red dot) moves along its locus, indicated as a green curve. A few frames of the animation are given below:

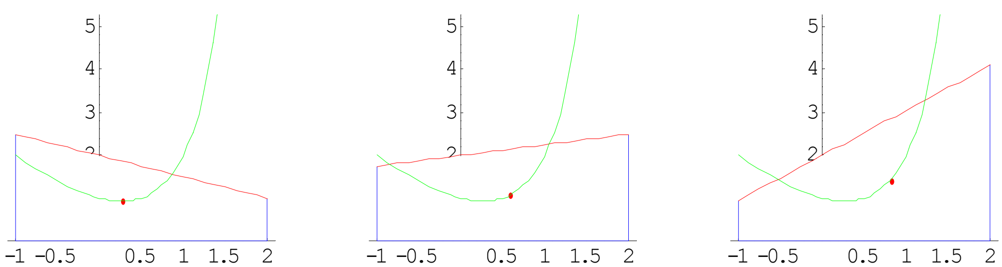

Figure 4.1 An Animation of the Center of Gravity of a Trapezoidal Region

(b) *Variable Parabolic Regions*

Let us reconsider the parabolic region $\mathcal{S}$ described in section 3(a). Suppose now that we keep the constants *a*, *b*, *q* fixed, but change the slope *p* of the upper boundary. Then the center of gravity of the region $\mathcal{S}$ changes, and we want to study its locus. The equation of the locus can be found by eliminating p between the equations (3.3) and (3.4). Our findings are summarized below:

Theorem 4.2 Suppose that *a*, *b*, and *q* are fixed real constants with $a > 0$, $b < q$, and let *p* be real parameter. Let $\mathcal{S}$ be region bounded by the graphs of $f(x) = px + q$ and $g(x) = ax^2 + b$. Then the locus of the center of gravity $G(\bar{x}, \bar{y})$ of the region $\mathcal{S}$ is another parabola given by the equation

$$y = 8ax^2 + \frac{(2b + 3q)}{5} \tag{4.3}$$

Proof. Details are left to the reader.

One can make an animation of the center of gravity of the parabolic region $\mathcal{S}$ by modifying program 4.1. One can also discuss variable exponential regions and their centers of gravity by extending section 3(b) of the paper. However, due to the space limitations of the paper we will suppress the details.

5. Ruler and Compass Constructions of the Center of Gravity of Fixed Plane Regions

In the final section of the paper, we will consider how to construct the center of gravity of some of the regions discussed in sections 2 and 3. First consider the trapezoidal region $\mathcal{R}$ discussed in section 2. The validity of following geometric construction of the center of gravity G of the region $\mathcal{R}$ can be verified by using the formulas (2.1) and (2.2). The motivation came from dividing the trapezoid into a rectangle and a triangle.

Construction 5.1 Consider the trapezoid $ABCD$ as given in figure 5.1. Draw the line DE parallel to AB meeting BC at E. Let F be the midpoint of the line segment CE, and H be the midpoint of the line segment DE. Let I be the point of intersection of the medians CH and DF of the triangle CDE. Let J be the point of intersection of the diagonals AE and BD of the rectangle $ABED$. Then the center of gravity of the trapezoid $ABCD$ is the point G on the line segment IJ such that $IG : GJ = BE : EF$. Therefore, to divide the line segment IJ into the ratio $BE : EF$, proceed as follows: Through B, draw an arbitrary line ℓ, and on this line pick a point K such that $BK=IJ$. Then draw a line through E parallel to FK meeting ℓ at G'. Finally, find the point G on IJ such that $IG = BG'$. Then G is the center of gravity of the trapezoid $ABCD$.

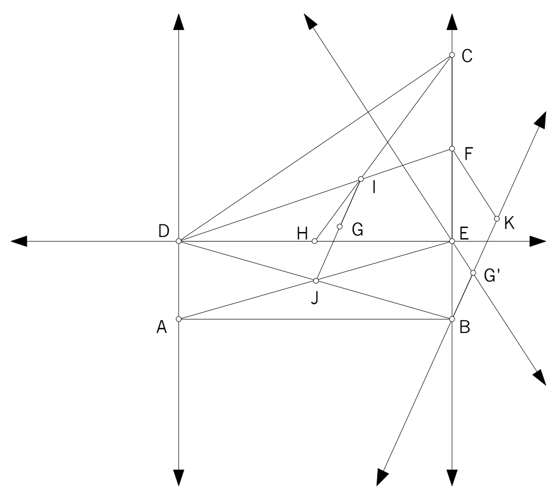

Figure 5.1 The Construction of the Center of Gravity of the Trapezoid $ABCD$

It is a more interesting problem to construct the center of gravity of the parabolic region $\mathcal{S}$ defined in section 3. Due to space limitations we are unable to include the details here.

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