Stimulating Conceptual Learning of Differentiation: A Graphic Calculator Integrated Curriculum

Ye Yoon Hong & Michael O. J. Thomas
The University of Auckland

One of the key aspects in a successful integration of technology in the secondary mathematics classroom is the inclusion of the classroom teacher in the process. Ideally this process should involve teachers in all stages, including planning, writing, and implementation. This paper describes the initial stages of a research project where a group of four committed and mostly experienced teachers in a secondary school are being mentored by a research team as they begin a year long programme to integrate the use of calculators (TI–89) with a computer algebra system (CAS) into their year 13 (age 17/18 years) mathematics curriculum. The students were given a procedural and conceptual test on differentiation prior to, and after completion of, the school module on differentiation. This module was taught using the calculators and the students had access to the calculators at all times during their learning. An analysis of the results of the test, discussion with the teachers, and observation of lessons suggest that while there appear to be some benefits from the use of the technology, the construction of an integrated approach to learning using a CAS which will assist inter-representational conceptual learning is not straightforward.

Background

In order to think about the constructs of mathematics we need some cognitive mechanism with which to examine them. The role of mathematical representations, both internal and external, is taking on increasing importance in mathematics education research. However, the idea of representation has a number of different uses in the literature. The description which resonates best with us is that of Kaput (1987, p. 23) who proposed that "any concept of representation must involve two related but functionally separate entities, the representing world and the represented world." In a later paper (Kaput, 1989, p. 169) he refers to a representation system as a correspondence between two notation systems (for example equations, graphs and tables of ordered pairs) co-ordinating the "syntax of one notation system with the structure of another.", and later still (Kaput, 1998) uses the terms representation system and notation system interchangeably. The concept of different representations of mathematical ideas also introduces an important class of mathematical activity involving "translations between notation systems, including the coordination of action across notation systems." (Kaput, 1992, p. 524). This involves manipulation of mathematical processes and concepts within a representation, and translations between different representations.

It is our contention that understanding of the properties and attributes of mathematical constructs, as well as procedures associated with them, are dependant on the form in which they are represented, and that student learning may be restricted by taking place within the limited confines of a single representation. For example students may fail to appreciate the full significance of eigenvectors in the vector field R^3 if they only encounter them in the algebraic or matrix representations, and never in the geometric one. The implication is that the construction of a full perspective of a mathematical concept by a student of mathematics means that they must engage with the concept in as many different representations of it as possible, forming links between the representations and becoming familiar with the actions that each representation facilitates. In this way they can be assisted to build *representational versatility*, a term we use (Thomas & Hong, 2001a, b) to encompass the ability to translate conceptual facets across representational boundaries (called *representational fluency* by Lesh, 2000 and similar to the *representational competence* of Shafrir, 1999) as well as the ability to interact with representations in both procedural and conceptual ways. For some concepts interaction with their representations in a conceptual way will require an object perspective of the concept rather than a process one. This process/object dichotomy has been described as involving the distinction between the dynamic process and static object view of mathematical concepts, which Sfard (1991), calls an operational and structural duality, as well as the manner in which the former is transformed into the latter in the mind of the learner. Sfard (1991) proposes that processes are *interiorised* and then *reified* into objects, while Dubinsky and his colleagues (Dubinsky, 1991, Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996) prefer to talk about processes being *encapsulated* as objects imbedding this in a theory of Action-Process-Object-Schema or APOS, for the construction of conceptual mathematical schemas. The algebraic symbolisation of a concept which can be perceived as either a process or an object has been termed a *procept* by Gray & Tall (1994).

One avenue which we consider holds considerable promise for encouraging students' *representational versatility* is the integration in learning schemes of graphic calculators, including those with built-in computer algebra systems (CAS). The value of these calculators is that they employ a number of linked mathematical representation systems, such as tables, algebraic symbols, graphs and ordered pairs, doing so in a way that provides a dynamic environment with instant feedback. The graphic calculator (GC), has already been shown to be capable of supporting the construction of mathematical meanings across representations (Kaput, 1989). Asp, Dowsey, and Stacey (1993), for example, reported a significant improvement in graphical interpretation and the matching of shapes with symbolic algebraic forms following the use of GCs for quadratic function graphing. Ruthven (1990) too has demonstrated how using the multi-representational features of GCs can help students link graphic and algebraic representations. They were assisted to recognise when a given graph came from a family of curves, and to build enriched problem solving strategies. Similarly a study by Harskamp, Shure, & Van Streun (2000) showed that access to GCs improved students function graphing approaches. However the research of Gray and Thomas (2001) reported little benefit from their attempts to employ GCs to help students link different representational forms.

One of the key discussions taking place around the use of GCs and other technology in the classroom is the way in which they might best be employed. Kissane *et al.* (1996) argue strongly that to be effective, GCs need to be fully integrated into all aspects of the mathematics curriculum, including assessment. Similarly, Leigh-Lancaster (2000) maintains that what he describes as the *congruency* between curriculum, pedagogy, assessment and values is critical in the effective use of computer algebra systems (CAS) in mathematics education. It has been suggested that such integration is the best way to open up the possibility of novel approaches to old topics, improvements in the order of presentation of topics, and a shift towards a better use of visualisation in the teaching some mathematical concepts (Ruthven, 1996, Kissane, 2000). However, while these may seem good arguments for curriculum reform (Harskamp *et al.,* 2000) there is an essential element in the process that must not be overlooked. Without the support and input of teachers any such changes are doomed to fail, along with other curriculum initiatives imposed without consultation. This research study recognises the crucial role of teachers and seeks to engage them in the process of investigating ways in which to integrate the CAS calculators into the curriculum and to analyse the potential benefits and difficulties of doing so.

Method

The Mathematics Education Unit at the University of Auckland has developed good relations with the teachers at a high performing Auckland girls' school over a number of years and we were asked to assist in the process of integrating the use of CAS calculators (TI-89s) into the year 13 curriculum. The roles of each group were made clear during our initial meetings with the teachers at the school. While we would be researching the implementation and were available as mentors who would make suggestions, the decisions on the actual implementation of the technology rested with the school. The group of teachers involved were experienced and committed, and included the deputy principal (an ex-head of the department), the current head of the department, another ex-head of the department, and a fourth teacher. Each of the teachers had around thirty years experience in the teaching of mathematics and two of them had substantial experience of teaching with both computers and calculators. Only the teacher who had not been the head of the department had not previously used calculators much in her teaching. Thus this research used a single group case study methodology, which has the disadvantage of not controlling for variables such as teacher input, etc.

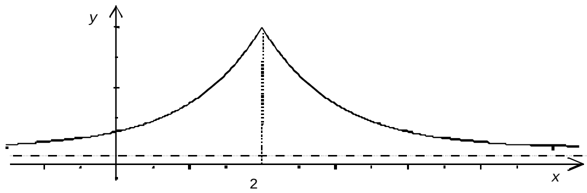
1. Differentiate each of the following functions with respect to x:

a) $y = 5x^5 - \frac{14}{x^3}$ b) $f(x) = \sqrt{2x-1}$

2. Differentiate $y = \frac{1}{2}x^2$ from first principles.
4. Find the value of the following limits:

a) $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4}$ b) $\lim_{x \to \infty} \frac{2x^3-3x+1}{5-x^3}$

5. Where is the function, whose graph is shown below, differentiable?



6. For the function $f(x) = x^3 - 3x^2 - 9x + 1$, find
   a) $f'(x)$ and $f''(x)$.
   b) the values of x for which the function has a maximum or minimum, distinguishing between them.
8. Let $f(x) = \frac{x^2}{x^2-1}$. For what values of x is $f(x)$ increasing?
9. Find $\frac{dy}{dx}$ if $x^2 - xy + y^2 = 1$. What is the gradient of the line perpendicular to the curve at the point (–1,1)?

Figure 1. Examples of the section A post-test questions.

The students were all girls aged 16 or 17 years and were in their last year of high school, with a university entrance examination at the end of the year. This is clearly a vital year of schooling for the students and often it is very difficult to get permission from schools to engage in research with such students. However, in this case since the school was driving the changes the problem did not arise. The students had all taken a basic introductory course in differentiation techniques in their year 12 classes and so had some familiarity with the concept of a derivative. The school had a set of 35 TI-89 calculators on loan and so each of the students was given their personal calculator to use for the year. They were thus able to use it at home as well as in lesson time.

1. Which of the following are polynomial functions? Give a reason.

$y = 3 - x$

Yes/No Reason________________

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$f(x) = 2x^4 - 3x^{\frac{3}{2}} + x$

Yes/No Reason________________________

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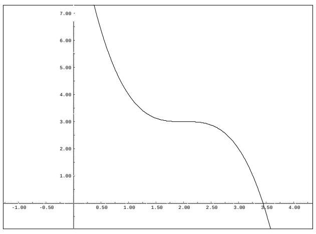
2. For the function shown below, $f'(2)=0$. Where is $f''(x)<0$?



4. On the axes below draw the graph of a function that is continuous everywhere and differentiable everywhere except x=2.

5. If $\frac{dy}{dx} = p^2$ and $x=f(p)$, where p is a parameter linking x and y, write down an expression for $\frac{dy}{dx}$.

7. The graph of $g(x)$ is obtained by translating the graph of $f(x)$ 2 units downward, i.e. parallel to the y-axis. If $f'(a) = -3$, what is $g'(a)$? Explain your answer.

8. Sketch the graph of a single function $y=f(x)$ where
   1. $f(x)\geq 0$ for all $x<0$ and $f(x)<0$ for all $x\geq 0$.
   2. $f(x)$ is not continuous at –1 but does have a limit.
   3. $f(x)$ is continuous at $x = 1$ but does not have a derivative.
   4. $f(0) = 0$.

9. For $y=f(x)$, $f'(2)=-2$. If the equation of the tangent to the graph of $y=f(x)$ at $x = 2$ is $y=mx+c$, what is the value of m?

10. If a given rate of change = $\lim_{x\to 0}\frac{\sqrt{2+h} - \sqrt{2}}{h}$, then:
   a) What is the function that we are calculating the rate of change of?
   b) At what point is the rate of change being calculated?

11. For what values of x is the function, whose graph is shown below, increasing?

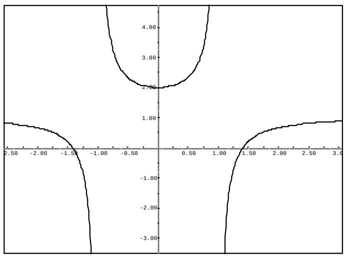


Figure 2. Examples of conceptual questions from section B of the post-test.

The school has a set scheme of work to cover and the research described in this paper is based on the second topic of the year, a five week long module called Calculus 1 – Differentiation. The scheme is very general in its scope and lists the university entrance examination prescription, the key skills to be covered (for example "find equations of tangents and normals"), textbook chapter references and suggested learning activities, one of which was "Use the TI 89 to investigate the rules of differentiation." While all the teachers had agreed to make substantial use of the calculators in their teaching there was considerable scope left for each teacher to integrate the use of the calculator into their normal teaching style. While this involved the students sitting in formal rows in the classrooms, the teacher circulated a lot and actively encouraged considerable classroom interaction.

Prior to the calculus teaching the students were given a test in two parts, A and B. The section A questions (see the examples in Figure 1) involved traditional skills such as differentiation techniques, finding limits, normals and tangents. It also included a question on the concepts of differentiability and increasing functions and, at the school's request since they used the results of this section for their internal grading, two application questions (these are not shown). These algorithmic questions were intended to give some idea of the influence of the use of the GCs on skills. In contrast section B (see Figure 2) was more concerned with questions aimed at conceptual understanding, to check whether the students were gaining an appreciation of the principles and not just algorithmic skills. The concepts addressed were: polynomials; increasing/decreasing; continuity; differentiability; parameter; conservation of gradient under translation; differentiation by first principles. The questions in section B also addressed the question of representational versatility. For example questions 2, 6, 7, 8, 9 and 10 all required the linking of algebraic and graphical forms, starting with one and converting to the other, while question 4 involved the graphical linking of two concepts. In addition section A question 8 and section B question 11 were deliberately linked in order to give some indication of representational versatility. The two questions are identical in content but differ in form, namely the representation used, with the section A question given algebraically with no graph and the section B graphically with no algebraic function. The students were able to use GCs during the test.

After the five weeks of teaching the students were given a parallel test (Figures 1 and 2 show post-test questions) comprising questions with minor variations in the content e.g. a translation in the y direction rather than the x direction in question B7. Once again they were able to user the GCs when taking the test since we were concerned with the influence of the calculators on method as well as performance. Of the students who took the module of work, only 28 completed both tests, and the results of these students are discussed below.

Results

What we were interested in was whether after using the calculators for the five weeks the students were better able to tackle the questions, especially those directed at concepts, than they were before. While it would appear surprising if they were not any better, especially on skills questions, since they had been taught the topics during this period, it has been experienced that students taught procedurally often do not learn the underlying concepts. We were also keen to discern whether there were any areas where it appeared that the calculators had been of particular value in their learning, particularly with regard to representational versatility. A limitation of the study was that we were unable to interview the students and so we were left having to infer their representational progress from their solutions to the questions in the test. This was not ideal but it was not possible to interrupt further the students' lessons.

After their work the students did better both on the section A questions (Pre-test mean=5.7, Post-test mean=14.2, t=10.6, p<0.0001) and the section B questions (Pre-test mean=0.43, Post-test mean=5.1, t=5.59, p<0.0001), and overall, than they did before. When looking for evidence of their representational versatility we found that in question B5 a number of students showed a preference for using $\frac{dx}{dp}$ rather than $f'(p)$ when differentiating $y = f(p)$. This was probably due to the need to use a version of the chain rule in the question, but shows a lack of versatility and the strong role that context plays in the selection of representations.

While students gave considerable evidence of visual thinking in their solutions, it was usually difficult to attribute this directly to the use of the GCs. Some of the thinking was good, and showed representational versatility, but some gave evidence of a limited perspective. For example, several students when asked to sketch a function in question B4 and B8 drew a linear or a quadratic graph, along the lines of that seen in Figure 3. This could be seen as either a good feature, in that they were drawing the simplest solution, or not so good in that they had not considered other, more general, possibilities.

Figure 3. An example of a quadratic solution to a general problem.

Another student, see Figure 4, had the correct link to the graphical representation in question B7 but still managed to get the wrong answer. She was unable to infer from the graphs that the gradient would stay the same. Thus representational versatility was lacking, even though she had a good visual sense.

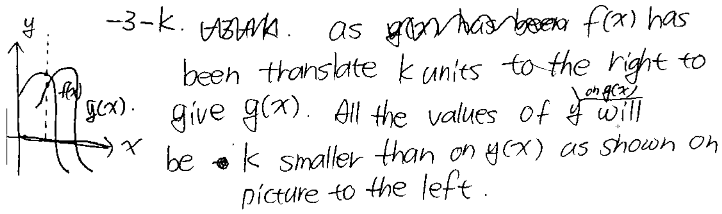


Figure 4. Using a correct graphical representation without success.

In contrast, two students (see Figure 5) were highly successful in B7 by this method and were able to answer the question by linking the graphical and algebraic representations in a powerful way, perceiving the function graphs as objects and being able to operate on them.

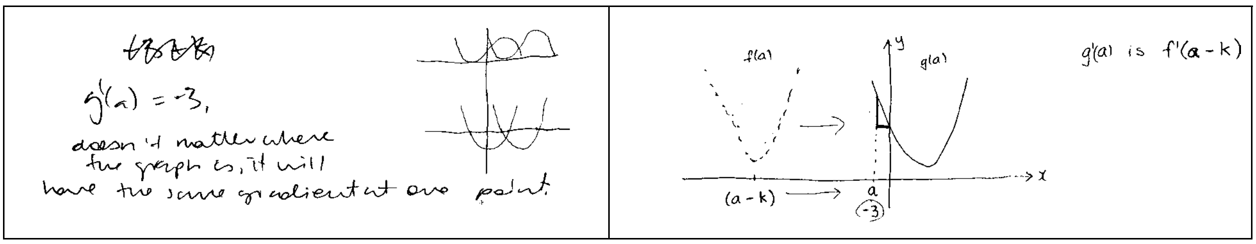


Figure 5. Good use of graphical representations.

Evidence of use of the calculator was sparse, and especially positive use, and this was a little surprising. One student seemed to have used the GC to check the value of a limit (question 4A) by using values of x=1000, and 2000 (Figure 6), while another used the GC to draw a graph to make the representational link between the algebraic calculation of maximum and minimum points and the picture (question 6A; see Figure 7).

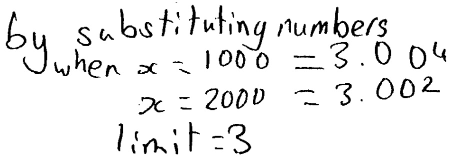


Figure 6. Using the GC to find a limnit.

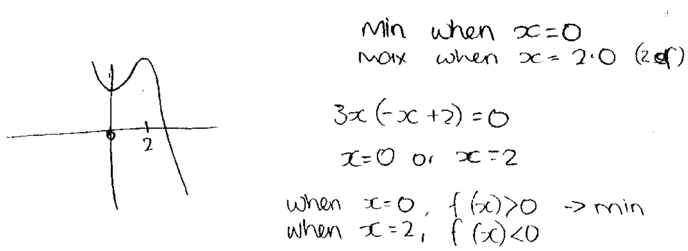


Figure 7. Using the GC to link graphical and algebraic representations.

The comparison between question 8 of section A and question 11 in section B proved interesting in terms of the representations. In the pre-test the graphical version (Q11) was answered significantly better than the symbolically presented version (Q8) (Q8 mean=0.04, Q11 mean=0.48, t=4.29, p<0.0005). When the post-test gain scores were analysed they showed that the gain on the symbolic question was significantly greater than that on the graphical question (Q8 mean gain=1.16, Q11 mean gain=0.68, t=2.17, p<0.05). At first sight this result is slightly surprising since it seems to indicate that students have improved more on the algebraic version of the question than on the graphical one, where the increasing nature of the function should have been more readily apparent. Most of the students had been taught to find where a function is increasing by a procedure based in the algebraic representation, namely using $f'(x) > 0$, and they learned and executed this quite well. However, the fact that they did less well in the graphical question shows, surprisingly, that the concept was not established in an inter-representational way. This could have been addressed in the teaching of the topic by using the CAS calculators to reinforce the concept across the representational boundary. They could have been encouraged to enter functions such as $f(x) = \frac{x^2}{x^2 - 1}$ into the calculator and then use it to draw the graph. Employing the trace facility would enable them to look along the curve's branches to see where the values of the function were increasing. This would then help them to answer problems like Q11 where although the graph was given, no formula for the function was attached to it and so the algebraic method could not be utilised. This shows one way in which the calculator could be employed. However, the fact that they could not cope as well with the graphical representation is evidence of a lack of representational versatility, and thus a weakness in the students' understanding of the concept of an increasing function.

One surprising weakness that had emerged prior to the study and for which we wanted to ascertain the influence of the GCs, was the ability of students to link the concept expressed by natural language representation ‘polynomial’ with the algebraic representation of this concept.

Question B1 asked whether $f(x) = 2x^4 - 3x^{\frac{3}{2}} + x$ and $y = 3 - x$ were polynomials, giving reasons (see Figure 2). Only two of the students answered either part this question correctly in the pre–test, and two in the post test (7.1%). However, the interesting thing was, not that the students did not attempt the question because they had no conception of polynomials, but rather that they thought they understood it, but were wrong. Among the wrong conceptions were:

Table 1
Answers to the question B1 on the definition of a polynomial

Student	$y = 3 - x$	$f(x) = 2x^4 - 3x^{\frac{3}{2}} + x$
A	No: Does not have a positive integer.	Yes: x has the positive integer ie) x^4 and $x^{3/2}$
B	Yes: both are the power of a whole number eg $3^1 – x^1$.	No: Because this is not a power of a whole number $–3x^{3/2}$.
C	No: x is not to a power of more than 1.	Yes: x is to a higher power than 1. 3
D	No: As x doesn't have power more than 1.	Yes: As x has power > than 1. 2
E	No: There are two variables.	Yes: the function is in relation to one variable.
F	No: *b/c* [because] it is a linear function.	Yes: *b/c* it is a graph. 2
G	No: Only one x term.	Yes: more than one x term. 6
H	No: There is [sic] no powers.	Yes: There is [sic] powers. 2
I	No: Not all functions of x.	Yes: All functions of x.

It is of some concern that 6 students thought that a polynomial had to have more than one term, since this seems to imply that they had been taught this, or certainly had learned it. Five of the students thought that a polynomial had to have powers of the variable greater than 1, and one that a polynomial could not have a constant term. Their answers also show a misunderstanding of a number of concepts related to polynomial in both algebraic and language representations: x has no power or no positive power; an integer is any positive number; 3 is a variable; and a linear function is not a polynomial. It would appear that the use of the GCs had had little or no effect on understanding of this concept. This simple example is a sobering reminder of the importance of communication in the classroom, and that we should never take for granted that our students are using words, or symbols or other representations in the same way that we as teachers are.

In summary we can say that there was very little evidence in the tests of successful and positive use of the GCs in the students' working in spite of the emphasis placed on them by the teachers. This may be because of a lack of time in the tests or a disinclination on the part of the students. There is also a sensitivity on the part of students to the context of a given representation which affects the form used and the meaning attributed to it. This does not mean of course that the GCs have not been helpful in the construction of the students' learning, which is altogether a different proposition from the assessment presented here. However, it is also clear that there are still a number of areas where these students need to build better conceptions and to improve their representational versatility (Thomas & Hong, 2001a, b), and further aspects of this study will be looking closely at this understanding. It appears that Leigh-Lancaster (2000) is correct in saying that a congruency including clear pedagogical direction when integrating the use of GCs into the senior school curriculum is essential. To obtain good results with learning, even using well trained and enthusiastic teachers is not necessarily an easy task.

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