Representational Fluency and Symbolisation of Derivative

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The nature of mathematical concepts has been the subject of some scrutiny in the mathematics education literature. One of the key ideas described in the literature is the distinction between the process and object perspectives of concepts. What is not so clear is how this distinction translates into different symbolisations and representations of the concept, and how students can construct the versatility to be able both to interact with these, and to translate between them. This paper describes the initial stages of a study which aims to use the dynamic multiple representations of a calculator's computer algebra system (CAS) to promote improved understanding of differentiation. In particular it looks at student interpretation of the symbolisation $\frac{dy}{dx}$ and their ability to relate this across different representations. The results of the study show that the students surveyed lack representational fluency for the concepts associated with this symbolisation, and that this appears to be due to their process perspective of the symbol. They have not yet encapsulated the differentiation process, as represented by the symbol, into the derivative object. Some implications for teaching and learning which can address this using the CAS calculator are considered.

Background

Symbolisation is a process which is at the very core of all learning, and especially so in mathematics. It involves forming a correspondence between two worlds, the one being symbolised and the one containing the symbols. The former may be physical or abstract, and the latter may be internal or external. Some authors, such as von Glasersfeld (1987) and Cobb (2000) have used the idea of symbolic representation to refer to *externally* written or spoken symbols which stand for something perceived as an experiential item; a segment of experience that has been sifted from the rest of experiences as a distinguishable piece, external, concrete entities, such as a mark on paper, or a display on a computer screen (or any technological device), or an arrangement of physical materials interpreted, standing for, or signifying, something else. However, symbols seem much broader in compass than this. When we see a physical shape we may interpret it as a particular mathematical shape using symbolism to link it with internal, mental imagery comprising the word square, or a prototypical image of a square. Symbolisation can thus be seen as involving the appropriation of meanings, between constructed cognitive structures, and between these structures and their external representational forms.

The way human minds try to understand their world of experience is through the construction of models which lead them toward understanding, or construction of a mental representation of a model (Moreno and Sacristan, 1995). When we are confronted with mathematical problems, we exploit visual and mental images, subsequently coding them into symbolic forms or representations. We may introduce language, symbols, graphs and organizational schemes that focus on hypothesised relationships, patterns and regularities attributed to the problem at hand (Lesh, 1999). Thus, according to Kaput (1987), a represented world emanates from an individual's interaction with the environment, producing mental events, which when acted upon by the mind produces mental/cognitive structures. In essence, the represented world is a cognitive construction of the individual as he interacts with the environment; an internal representation of the individual of his experience with this environment. This mental representation refers to the internal schemata or frames of reference used to interact with the external world (Dreyfus, 1991). Considering mathematics in this way as comprising internal mental operations, implies that meaning evolves through association of those mental operations with mathematical symbols (Lerman, 1994). This association may then be re-expressed and translated (re-presented) for the purpose of communication (Noss and Hoyles, 1996; Yackel, 2000). A symbol or notation system provides a means by which one represent a mathematical structure is sometimes called a representation system (Kaput, 1998), and the words have been used interchangeably. The correspondence and the relationships between the represented and the representing worlds is defined by Goldin (1987) as symbolisation.

Student interaction with symbols

Every representational system uses symbols, and rules to connect those symbols. The establishment of this correspondence, however, is not free from ambiguity (Goldin, 1987). In the classroom context where students' experiences differ, interpretation of a given symbol may come in a variety of ways. When a new symbol is introduced by the teacher as a new notation for a mathematical concept, students try to assimilate it to their existing schemas, which may bring clusters of templates where it may fit, evoking meaning within available schemas derived from individual experiences. It is when the students enter into a discourse among themselves, or with the teacher, that the meaning constituted by the symbol is appropriated, through negotiation. In classroom situations, the negotiation of meaning between the teacher and the students is necessary, as the former directs the students to the taken-as-shared understanding of the symbol, together with its meaning. As Sfard (2000, p. 66) says, Òconversational feedback plays a central role in discursive and experiential background for the introduction of the sign.Ó

This research considers the algebraic symbol rich area of differentiation. Here symbols such as $f'(x)$or $\frac{dy}{dx}$ may be interpreted in two quite different ways. They may be understood as a process (of differentiation), where a function has to be differentiated with respect to a variable x, or they may be interpreted as an object, the derivative of the function with respect to x, that is the derived function. These two perspectives are related of course, and the differentiation process needs to be encapsulated as the derivative object in order for one to be able to operate further on the resulting function. This encapsulation of processes as objects occurs in many places in mathematics and has been the subject of considerable analysis (Dubinsky, 1991; Tall, 2000). Further, the combination of algebraic symbols, evoking a process, and/or a concept has been defined by Gray and Tall (1994) as a *procept*. This process/object (or process/concept) duality of symbol use draws attention to a better and deeper understanding of mathematical objects and the process by which concepts are built and represented (Graham and Thomas, 2000). Hence, what may at first appear to be an ambiguity of symbol use is in fact part of the requirement of versatility of symbol interpretation which arises in many parts of mathematics. Such richness of interpretation of symbol use is not automatic though and needs to be constructed by the individual, and an analysis of this process is one aim of this study.

Proceptual understanding finds an analogue in the interpretation of the function of symbols as signs or counterparts (Pimm, 1995). As a sign, the symbol names or points to something else, drawing attention away from the symbol to what it represents. When a symbol acts as a counterpart, an actual relation or connection exists between the object and its symbol. This latter perspective leads to its manipulation — one acts on it and treats it as if it were the object. In contrast, the use of symbol as a sign directs one to what it stands for, or to what it represents. Applying this to the symbol $\frac{dy}{dx}$, we can say that its use as a counterpart points one, via the use of the variables y and x, to the process of differentiation, or at least to a procedure for it to be applied. Functioning as a sign, however, $\frac{dy}{dx}$ may be seen as representing either a derived function, or a gradient (of a tangent), or a rate of change; ideas that have nothing directly in common with the symbol.

Representational versatility

The meaning of mathematical constructs, rather than existing in a single representation system, tends to be distributed across several interacting representational systems, with each emphasising and de-emphasising different characteristics of the constructs (Lesh, 1999, 2000). The ideas grow in various modes and forms of representation with each supporting an aspect of mathematical thinking (Tall, 1994). Thus, making multiple representations available can tend to improve the capacity for learning, provided the links between them are addressed (Kaput, 1992; Noss and Hoyles, 1996). In particular, since each representational form captures a special aspect of the concept, equivalence of meaning between any two representational forms must be established. This ability to establish meaningful links between and among representational forms and to translate from one representation to another has been referred to as *representational fluency* (Lesh, 1999), or as *representational competence* (Shafrir, 1999). Thomas and Hong (2001a, b) have introduced the concept of *representational versatility* to include both fluency of translation between representations, and the ability to interact procedurally and conceptually with individual representations.

One avenue which we consider holds promise of encouraging students' *representational versatility* is the integration in learning schemes of graphic calculators, including those with built-in computer algebra systems (CAS). These calculators not only employ a number of linked mathematical representation systems, such as tables, algebraic symbols, graphs and ordered pairs, but they do so in a way that provides a dynamic environment with instant feedback. Their forerunner, the graphic calculator (GC), has for some time been suggested as capable of supporting the construction of mathematical meanings across representations (Kaput, 1989). Ruthven (1990) demonstrated that this could be the case by using GCs to help students link graphic and algebraic representations. They were assisted to recognise when a given graph came from a family of curves, and the multi-representational features of the GC were shown to enrich problem solving strategies. One study with students on the influence of GCs on quadratic function graphing (Asp, Dowsey, and Stacey, 1993) reported a significant improvement on interpreting graphs and matching their shape with symbolic algebraic forms. Another by Harskamp, Shure, & Van Streun (2000) showed that access to GCs improved students function graphing approaches. However the research of Gray and Thomas (2001) reported mixed results on their efforts to employ GCs to help students link different representational forms.

The above discussion supports the contention that in order to understand how students build understanding of derivatives and differentiation, it is important to investigate their representational abilities. The research reported on here forms part of a much larger study investigating how the use of CAS graphic calculators might enhance student conceptual understanding of differentiation, and records the results of a pilot study investigating the representational fluency of students. In particular it aims to assess the role of students' concept representation in understanding of differentiation and derivatives, addressing the following questions:

(i) How do the students understand symbols such as $f'(x)$ or $\frac{dy}{dx}$?

(ii) What concepts do they attach with the use of $f'(x)$ or $\frac{dy}{dx}$ in different representational contexts?

(iii) What representational translations involving differentiation concepts are students capable of?

(iv) What representational forms do they use in solving problems involving derivatives?

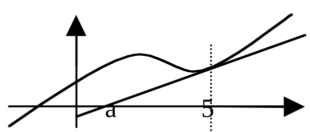
Method

In order to ascertain how we might advantageously utilise the CAS calculators in the multi-representational learning of differentiation we first need to know what level of representational versatility students have.

1. If $y=f(x)$ is a function, which of the following are functions? Give reasons.

(i) $f'(x)$ (ii) $\frac{dy}{dx}+f''(x)$ (iii) $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

4. Write the following in symbols:
   (i) The gradient of a function *f(x)* changes at the rate *x*.
   (ii) For a function *y*, $\frac{dy}{dx}$ changes at the same rate as *y*.
   (iii) The gradient of a function $y = x^5$ changes at the rate *x*. Write this in terms of *x* only.
5. Explain in words what the following notations mean to you.
   (a) $\frac{dy}{dx}=k$, where *k* is a constant
   (b) $\frac{d^2y}{dx^2}=m$, where *m* is a constant
   (c) $\frac{d(\frac{dA}{dp})}{dp}=\frac{dA}{dp}$
6. If $f'(5)=1$ and $f(5)=3a$, use this information, and that on the graph of *f(x)* below, to find the value of *a*. [The given line is a tangent to the graph at *x*=5]



7. For the function *g(t)* shown in the table, write down the average rate of change of *g(t)* from *t* = 0 to *t* = 1.

t	*g(t)*
0	0
1	*k*
2	3
3	16
4	47

10. What do you understand $\frac{dy}{dx}$ to stand for/represent in each of the following?

a) Rate of change of *y* wrt *x* b) The gradient of the tangent of *y=f(x)* c) The derivative of *y* wrt *x* d) A differential e) A term in an equation f) None g) Other: _______________

(i) $\frac{dy}{dx} = 3$ (ii) $\frac{dy}{dx}=5x$ (iii) $\frac{dy}{dx}=4y$ (iv) $z=\frac{d(\frac{dy}{dx})}{dx}$ (v) $\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$ (vi) $2x+\frac{dy}{dx}=1$

Note: The format of some questions has been changed here (e.g. space for writing answers) to save space.

Figure 1. Examples of the test questions.

To gauge this a test was constructed and administered to thirty-two Year 13 students (aged 16 or 17 years) in a top-performing all-girl school in Auckland, New Zealand. It was made clear to the students that the test results would not be included in the school's assessment of their performance. The students had all studied basic derivatives and differentiation techniques in Year 12, and the test was given after their first Year 13 differentiation module. It was designed to probe student understanding of derivatives and their ability to translate between representational forms. In order to investigate how students understand and represent derivatives in various representational forms, and to account for their representational versatility, problems were designed which required students to translate from one representational form to another (see Figure 1 for examples of questions), including translations from words to symbols and symbols to words. Other problems included the use of algebraic, graphical and tabular representations with the aim of investigating how students understand them and whether they could translate the given representational forms into others.

As can be seen in Figure 1, question 10 gave some examples of basic equations containing $\frac{dy}{dx}$ in which it might be understood in subtly different ways, and students were asked to classify each as either a rate of change, derivative, gradient, differential or as a term in an equation. Our expectation was to check whether students could see that while the notation can have different interpretations, in the absence of a context any of them could be appropriate. Question 4 addressed the symbolisation of rate of change of gradient and question 5 asked for an interpretation of algebraic symbols, including one which requires appreciation of the use of $\frac{dA}{dp}$ to represent an object rather than a process. Question 6 looked at the linking of $f'(x)$ with a graphical representation and question 7 rate of change with a tabular one.

Results

Of the thirty-two students taking the test, twenty-two students either completely or partially answered the test, while a further six students did not write anything on their papers. When the school was questioned about this it became clear that a small number of students had taken a less than serious approach to completing the questionnaire, not bothering to fill anything in. While this was disappointing, it had been made clear that the test was entirely voluntary and would not count towards their assessment. The analysis and interpretation presented below of the results of the 22 responses is based on representational versatility, that is the interaction with and translations exhibited by the students. These interactions and translations, whether explicit or incidental were identified and classified.

$\frac{dy}{dx}$: Derivative, rate of change and gradient

Table 1 gives a summary of the interpretations of the use of $\frac{dy}{dx}$ in question 10 (see Figure 1). It was evident in students' translation and interpretation of the symbols that they had engaged with the use of the symbol $\frac{dy}{dx}$ both as a derivative of *y* with respect to *x* (*y* wrt *x*), as a rate of change, and as the gradient of a tangent of $y = f(x)$. However, interestingly, the students, gave quite varied interpretations of the meaning of $\frac{dy}{dx}$ depending on its contextual use. For example, the number of students who interpreted the use of $\frac{dy}{dx}$ in $\frac{dy}{dx} = 3$ as a rate of change of *y* wrt *x* slightly increased when the equation for which it was used was changed to $\frac{dy}{dx} = 5x$ (from 14 to 16). But when the right-hand-side of the equation involved the *y* variable, as in $\frac{dy}{dx} = 4y$, there was a drop in this (χ^2=3.28, *n.s*) with only 7 students classifying $\frac{dy}{dx}$ as a rate of change of *y* wrt *x*. However, when $\frac{dy}{dx}$ was used in the equation $2x + \frac{dy}{dx} = 1$, there was a significant drop (to 3, $\chi^2 = 9.59$, $p<0.05$) in the number of students who viewed it as a rate of change of *y* wrt *x*, and this further reduced to 1 in the other two cases.

Table 1
Student interpretations of $\frac{dy}{dx}$

Expression/ interpretation	Rate of change	Gradient of tangent	Derivative	Differential	Term of an equation
$\frac{dy}{dx} = 3$	14	10	14	0	1
$\frac{dy}{dx}=5x$	16	6	11	0	2
$\frac{dy}{dx}=4y$	7	4	7	4	1
$z = \frac{d(\frac{dy}{dx})}{dx}$	1	1	0	5	3
$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$	1	0	0	5	4
$2x+\frac{dy}{dx}=1$	3	0	5	0	8

A very similar pattern can be seen in the interpretation of $\frac{dy}{dx}$ as a gradient of tangent to $y = f(x)$, and as a derivative. In the expression $\frac{dy}{dx} = 3$, 10 students responded that $\frac{dy}{dx}$ represents a gradient; with $\frac{dy}{dx} = 5x$, there were 6; for $\frac{dy}{dx} = 4y$, four; but for the equation $2x + \frac{dy}{dx} = 1$, no one chose the option of $\frac{dy}{dx}$ representing gradient. Likewise with regard to interpreting $\frac{dy}{dx}$ as the derivative of *y* wrt *x*, the number of corresponding responses decreased similarly (see Table 1).

The pattern of answers for interpreting $\frac{dy}{dx}$ as a term in an equation was quite different. The number of corresponding responses increased rather than decreased. This provides further evidence of a well-known case in the literature that students often interpret the use of an equals sign as a signal to do something and produce an answer (Thomas, 1994). However, there is more than that here. The tendency to view an equation in this way seems to decrease as the complexity of it increases. In the first three equations virtually none of the students saw them as equations, probably because the $\frac{dy}{dx}$ appears alone on the left hand side, leading them to the process and result interpretation of the symbols. In the equations where they leave this interpretation the $\frac{dy}{dx}$ is no longer singular and in two cases appears on the right.

The distinction between $\frac{dy}{dx}=5x$ and $2x+\frac{dy}{dx}=1$ here is a particularly interesting one, given that both equations can have the form $\frac{dy}{dx}$ = *f(x)* where *f(x)* is linear. 72.7% see $\frac{dy}{dx}$ in the first of these as standing for a rate of change, but this drops to just 13.6% for the second. On the other hand, only 9.1% see $\frac{dy}{dx}$ in the first case as a term in an equation, but this increases to 36.4% for the second. All of this seems to be clear evidence that the role of symbols is very subtle and we need to pay close attention to the representational systems in which students meet various symbolisations in differentiation and encourage them to make links which can increase their representational fluency.

Translation from algebraic symbols to words

When asked to explain in words what the notation $\frac{dy}{dx}=k$*; where k is a constant* meant, the students' responses were as follows: a gradient (7), a derivative (3), or a rate of change (2). This was in contrast with the results of question 10, where a similar format with *k*=5, produced 14 choosing each of rate of change and derivative, and only 10 selecting gradient. Again their choices seem to be extremely sensitive to small changes in the context of the symbols. Some students provided graphical interpretation for the symbols describing: the Ò*graph is a straight line*Ó, or Ò*constant function — eg.a straight line where m is always the same*,Ó or Ò*this is a straight — gradient constant at k (straight, horizontal line)*.Ó Though these interpretations might not yield correct answers, they do indicate that the students had created links between graphs and algebraic symbols.

Similarly, when the students were asked to describe in words $\frac{d^2y}{dx^2}=m$, where *m* is a constant, they explained it as the gradient of the gradient/tangent (3), or the rate of change of gradient/change (3), or the second derivative is *m* (5). One student interpreted the symbols as a process (or procedure) indicating it was symbol used Òto determine max or min points.Ó Two students seemingly gave a graphical interpretation writing that Òthe concavity of the function *y* doesn't changeÓ, and Òfunction = a parabola.Ó These latter answers again point to the strength of influence of the learning context on understanding of symbols.

With the symbols $\frac{d\left(\frac{dA}{dp}\right)}{dp}=\frac{dA}{dp}$, one student saw $\frac{dA}{dp}$ on the right-hand-side as different from its counterpart on the left-hand-side describing Ò*the rate of change of A in respect to p is the same as the rate of change of A's derivative in respect to p.*Ó It appears that the position of $\frac{dA}{dp}$ in its context changed its meaning for this student, from derivative on one side to a rate of change on the other. Two students classified the notation as the *second derivative of a parametric function* indicating lack of flexibility on the meaning attached to the use of letters, such as *p*. They linked it to a context where they had used this letter a lot, namely the parametric representation of function in co-ordinate geometry. One student also translated the equation to another symbol form to interpret it, writing, $f''(x)=f'(x)$ - *point of inflection.* In addition to introducing new symbols, it is evident that the student is assigning a graphical interpretation to the notation.

Translation from words to algebraic symbols

Translations from words to symbols were also required in answers to problems such as Ò*The rate of change of the area of the shape É is* $\frac{1}{10}$ *of the area A at any given time. Write this in* *symbols*.ÓA considerable number (13) of students used $\frac{dA}{dt}$. Similarly, with the problem Ò*The rate of change of the area A wrt r* É*s* $\frac{1}{10}$ of *r*. *Write this in symbols*,Óthey used $\frac{dA}{dr}$ (14 students). The students' answers indicates that in this case they were able to discriminate successfully between the phrases *rate of change* implying time, and *rate of change* wrt *r*, evoking in them the use of the symbols $\frac{dA}{dt}$ and $\frac{dA}{dr}$, respectively.

Translations involving other representations

Questions 6 and 7 both require the solver to be able to link across two or even three representations. In the first of these the students have to relate the algebraic form $f'(5) = 1$ with the gradient of the tangent at *x*=5 for the given graph. Since no formula was given for the function *f* it was not possible to find the gradient of the tangent at this point any other way. Only three students (13.6%) made this link, with only one going on and correctly solving the problem, while 17 students (77.3%) did not write anything at all. Question 7 on the other hand involved the linking of the words Ôaverage rate of changeÕof g with a tabular representation. We had considered that many students might have a reasonably clear concept of average rate of change graphically and wondered if this would translate to the table form. In the event 14 students (63.6%) wrote nothing in response to this question. There were 5 correct solutions (22.7%), two of whom needed to introduce a third representation, an algebraic one, into the problem. The rate of change was strongly linked in their schemas with an algebraic form of the derivative and they wrote $\frac{dg}{dt} = k$ (not, interestingly $g'(t) = k$). A third student also used $\frac{dg}{dt}$ but incorrectly obtained $\frac{1}{k}$ as the answer. The results on these two questions again confirm that the representational fluency of these students is not strong in this content area, and that our planned calculator research to address this could prove useful.

Conclusions

The results of this preliminary study confirm that students do indeed have different interpretations the concepts of derivative and differentiation depending on the representational system in which they are symbolised. The following summary outlines some of the key conclusions from the study of students' interpretation of representations:

- The interpretation of algebraic symbols such as $\frac{dy}{dx}$ in an equation varies depending on the context in which it is used. Further this interpretation is very sensitive to minor contextual changes, such as whether it is expressed as the subject of an equation or whether the right hand side is a constant or a variable.
- The influence of the differentiation context in which algebraic symbols are first encountered is strong. This confirms the need for students to experience a number of contexts and a number representational systems for each construct.
- There is a support for a *proceptual* interpretation (Gray and Tall, 1994) of the use of symbols such as $\frac{dy}{dx}$. Students see these either as representing a process (of differentiation) or as a concept (of derivative), depending on context and other factors.
- There is evidence to show that students do make use of other representations in making sense of the given representation in differentiation, especially the graphical form. These can be summarised by the following translational shifts in

representation: words to symbols, symbols to words, symbols to graphs, and symbols to symbols.

- The students, who are from a top performing school, showed weakness in their representational fluency when asked to translate concepts across unfamiliar representational boundaries.

While the results of this study cannot be generalised they do indicate that there is a need to develop greater representational versatility among students. There are many opportunities to use a CAS calculator to assist students to construct such fluency. For example, we found that students were more likely to view algebraic symbols such as $\frac{dy}{dx}$ as a gradient if it was equated with a constant. Using a dynamic CAS environment they could be helped to see gradient as dynamic rather than static, and so come to appreciate that it can be represented by a variable quantity. There seems no doubt that student use of symbol can be informative about how they perceive a problem situation, and thus provide guidelines in designing suitable interventions (Dufour-Janvier, 1987). In order to help students in their construction and development of mathematical concepts and their representation, the knowledge of their ability to translate from one representation to another provided here will serve as a useful instrument in the design of instructional modules and materials which use integrated CAS calculators to guide students in their representation-building processes. It is, however, important to underscore that it is the students who must do the work of building and revising their representations, and that the role of the teacher is to provide clear guidance and direction along with rich opportunities for successful construction. It is, therefore crucial that the teacher creates an environment that will enhance meaningful use of symbols and multi-representational aspects of concepts, and where there is access to the students' symbol use in relation to their understanding of mathematical concepts. It is our contention that CAS calculators can form a valuable part of such environments, as we hope our further research will confirm.

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