The transition from scientific calculators to computer algebra systems in one educational system.

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Abstract

The Danish Upper Secondary School Leaving Examination includes the 3-year A level mathematics course and, for the first time in 2001, schools could allow their students to complete their mathematics examinations with a CAS calculator. Access to a standard graphics calculators was assumed for 2000 and 2001, for those not using a CAS calculator. The introduction of CAS calculators into assessment was the culmination of 3 years of work by the Ministry of Education. (Vagner, 2001).

This investigation has shown that, despite the introduction of CAS, there has been little change to the question style in most cases, though there is evidence of the questions including more words to describe the mathematical context. It is also apparent, from looking at the examinations, there is a need for students to have a thorough technical understanding of the graphic calculator and CAS. There also appears to be difficulties with the differing functionalities of the CAS calculators that are currently available.

Danish National Mathematics Curriculum

The Danish National Curriculum for Gymnasiums is made up of three mathematics subjects

- Matematisk Linje 3-Årigt Forløb Til A-Niveau, (three year A level, for students studying the mathematics course)
- Matematisk Linje Og Sproglig Linje 2-Årigt Forløb Til B-Niveau, (two year B level mathematics course for students completing either the mathematics course or the language course)
- Matematisk Linje 1-Årigt Forløb Til A-Niveau, (a one year course for those wishing to transfer from the B level to A level)

The aims for the A level mathematics are as follows;

(a) The students should gain an understanding of a number of fundamental mathematical modes of thinking, concepts and methods.
(b) The students should become familiar with mathematics as a means of formulating, analysing and solving problems in various areas of the subject.
(c) The students should further develop their ability to use mathematical concepts and methods on their own, and they should become able to acquaint themselves with, analyse and evaluate problems that can be formulated and dealt with by means of mathematical concepts and methods. (Danish Ministry of Education, 2000c).

The course content for A level mathematics is made up of five main areas which are; Numbers, Geometry and vectors, Functions, Infinitesimal calculus, Statistics and probability

Assessment
From 2000 there were two written examinations for the A level. Prior to this time there was one written examination in which scientific or graphic calculators were allowed. In addition, there has always been an oral examination. Beginning in 2000 four hours were allotted to the first written examination. According to the Danish Ministry of Education (2000c). "Questions are set on central sections of the five main subject areas and are based on the assumption that the candidates have pocket calculators with graphic displays, Mathematical Formula Collection for the 3-year course for A-level (published by the Ministry of Education), as well as a collection of tables, including tables of binomial coefficients, cumulative binomial distributions and normal distribution." It was also assumed from 2000 that all students would have access to a graphic calculator though, there was no minimum specification for the functionality of the graphic calculator.

The second written examination is of two hours duration and, apart from the availability of graph paper, students are expected to complete the examination without a book of formulas, tables or calculators. (Danish Ministry of Education, 2000c).

Analysis of examinations.

To enable valid comparisons to be made, between the examinations in 1994 (prior to the introduction of graphic calculators as allowed calculators in 1995), and those in 2000 and 2001, topic areas were chosen that would most likely be impacted by the use of CAS in 2001.

Between 1997 and 2000 there were a number of experimental examinations held in conjunction with the normal A level mathematics examinations. These examinations were often very similar to the one undertaken by the majority of students. In 2000 there were two examinations, one for the regular A level classes and another examination for the experimental classes in three gymnasiums Fjerritslev, Fredriksborg and Haslev. (Danish Ministry of Education, 2000b) In each examination all students sat the paper without any aids at all (Danish Ministry of Education, 2000c). There were only minor variations between the questions on the CAS paper and the graphic calculator only paper (Vagner, 2001)

The percentage of the total mark allocation for the questions in each examination were;

Year	**Opgave (exercise) Numbers**	**Marks as a percentage of the total available**
Danish Ministry of Education (1994)	1, 4 and 6	50%
Danish Ministry of Education (2000a)	4, 6, 7a and 7b	50%
Danish Ministry of Education (2000b)	4, 6, 7a and 7b	50%

Functions

In 1994 there was no specific question on functions. In 2000 Opgave 7a, (one of the two questions which students were required to chose between), was specifically about functions. The graphic calculator and the CAS questions for 2000 were similar, a rational function was given and the domain and the equations of the asymptotes were to be found (Figure 1).

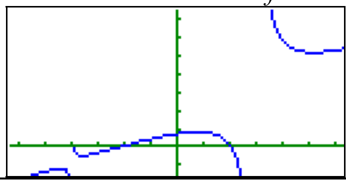
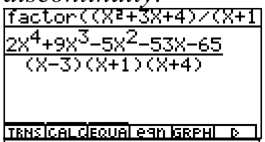
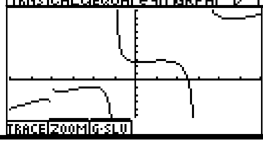
Question	Dividing and Factorising	Domain and asymptotes for f
Opgave 7 a. (Non CAS Graphic Calculator only) *A function f is defined by* $f(x)=\dfrac{x^3+4x^2-4x-17}{x^2+x-12}$ *Find the domain of f.* *Find an equation for each asymptote for the graph of f.* (Danish Ministry of Education, 2000a)	*Dividing the functions using pencil and paper techniques, will give* $=x+3+\dfrac{5x+19}{(x+4)(x-3)}$ *or using graphs and tables graphically the discontinuities can be found.* 	*Using this solution and or the graphical calculator and tables on the calculator students can determine the domain as* $x \in R/\{-4,3\}$ *Using the previous answer or graphical solution the asymptotes are found to be* *y = x + 3, x = - 4 and x = 3.*
Opgave 7 a. (CAS allowed) *A function f is defined by* $f(x)=\dfrac{x^2+3x+4}{x+1}+\dfrac{x^3+4x^2-4x-17}{x^2+x-12}$ *Find the domain of f.* *Find an equation for each asymptote for the graph of f.* (Danish Ministry of Education, 2000b)	*Combining the fractions and factorising the denominator gives the points of discontinuity.*  	*The domain of f is* $x \in R\setminus\{-4,-1,3\}$ *and the asymptote is* $y=2x+5$ *Depending on the CAS used, division of the function is used to determine the oblique asymptote.*

Figure 1 Questions and possible solution methods for the graphic calculator and CAS questions in 2000.

The table or graphing facility can assist students to find the points of discontinuity and therefore the domain in either the GC or the CAS question. Students using a CAS can factorise the denominator algebraically to help identify the discontinuities, but finding the oblique asymptote is problematic and dependent on the type of CAS being used. These two questions are very similar in nature, though the CAS question involves a more complex function. But the steps in the mathematical solution processes required to solve the problems are similar in each case and can be given as;

- Domain is found by considering the end behavior of the function as well as looking for points of discontinuity.
- Using the factorized form of the denominator to locate the vertical asymptotes
- The oblique asymptote is found by division using pencil and paper techniques when using the GC. Whereas in the CAS case it is possible to find the asymptote by selecting the appropriate function on some CAS calculators but not others.

The level of difficulty in each case would appear to be the same, though different skills may be required depending on the technology being used. Students need to have a similar level of mathematical understanding of the problem to solve it. That is the advantage of the CAS is that it completes the manipulations (and allows the students to focus on the solution process).

A real life problem using functions.

The following question appeared in both papers in 2000.

> *The number of types of plants on the individual islands of Galapagos can be estimated by a calculation based on the island's area x, measured in square miles and the function N(x). For the function N(x), it is known that N(15) = 68 and N(174) = 149. And that the equation for N is of the type* $N(x)=b\,x^a$.
>
> *Calculate the numbers a and b. Calculate the proportion between the number of types of plants on two different islands, where one has an area which is 2.5 times as big as the area of the other. (reference: Charles J Krebs: Ecology, New York 1972)*
>
> (Danish Ministry of Education, 2000a); (Danish Ministry of Education, 2000b)

The comparison of the mathematical solution process using a GC and a CAS for the question is given in Figure 2.

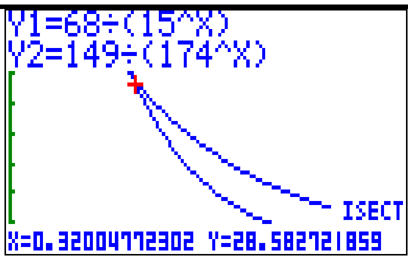
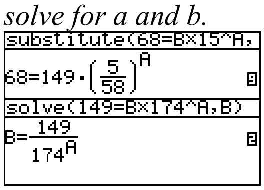
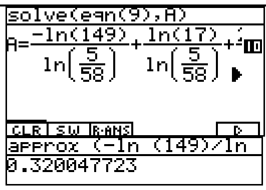
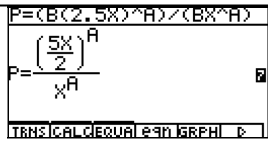
	Substitute for initial values	Solve for a and b	Find the proportion.
Graphic Calculator (P&P) solution	$N(x)=bx^a$ $N(15)=68=b(15)^a$ $N(174)=149=b(174)^a$ Rearranging the equations gives $b=\frac{68}{15^a}$ and $b=\frac{149}{174^a}$	 *a= 0.32 and b = 28.583*	where $A_1 = 2.5\,A_2$ $\frac{N(A_1)}{N(A_2)} = \frac{b(2.5\,A_2)^a}{b\,A_2^{\ a}}$ $= 2.5^a = 2.5^{0.32004772}$ $\frac{N(A_1)}{N(A_2)} = 1.3401$
CAS solution	*The CAS can be used to substitute into the equations to solve for a and b.* 	 *The use of approximation is required to give a numerical solution for a and b which is no different to the graphic calculator.*	 *Substituting for A will give the required proportion.*

Figure 2 Possible solution strategies for the graphic calculator and CAS question in 2000.

It is evident that there is minimal (if any) advantage for a student using a CAS, in both cases they need to develop a method of solution for the question, set up the equations and then solve as required. The CAS will mainly be used for manipulation of the algebraic equations while the graphic calculator will be used to find the solutions from a graphical perspective. The important steps in the solution process are:

- Setting up the equations.
- Recognise that substitution is an appropriate method to find *a* and *b*.
- The use of the general equations for area and a variable to connect the two areas giving a ratio from which the students can calculate the proportion.

Integral Calculus

Integral calculus questions appear in most end of high school calculus examinations. The use of a CAS has the potential to reduce the need to memorise many of the standard integration techniques, therefore, the impact on questions may be significant. All of the examinations considered, included a question requiring the calculation of the area and volume of a function that has been rotated about the first axis (*x*-axis) (Figure 3)

Danish Ministry of Education (1994)
Opgave 1

Two functions f and g are given by $f(x)=\frac{1}{2}x^2$ and $g(x)=2x$

The graphs of these two functions define an area M. Calculate the exact value of this area. Calculate the exact value of the volume of the body produced by rotating M 360^0 *around the first axis.*

Danish Ministry of Education (2000a)
Opgave 4

A function f is defined by

$$f(x)=x-2\sqrt{x}\,,x\geq 0$$

Calculate the zeros for f. Define the intervals of growth for f. Draw a graph for f and indicate the range. In the fourth quadrant the graph for f and the first axis define and area M. Calculate the area M using the definite integral. Calculate the volume of the body produced by rotating 360^0 *around the first axis by using definite integrals*. [Note that the use of the term “using the definite integral” implies an exact solution and therefore pencil and paper techniques are required.]

Danish Ministry of Education (2000b)
Opgave 4

A function f is defined by

$f_a(x)=x-a\sqrt{x}\,,a>0\,,x\geq 0$.

Calculate the zeros for f_2*. Define the intervals of growth for* f_2*.*

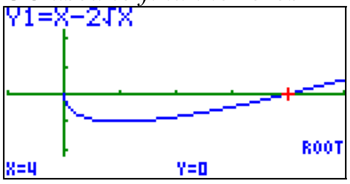
Draw a graph for f_2 *and indicate the range. In the fourth quadrant the graph for* f_a *and the first axis define and area* M_a*.* A_a *is the area of* M_a *and* V_a *is the volume of the body of rotation which is produced by rotating* M_a 360^0 *around the first axis. Calculate* A_2 *and* V_2

Calculate the value of a for which $V_a=\pi A_a$

Figure 3 Integral Calculus questions from 1994 and 2000

The solution process for each of the questions is given in Figure 4. The steps in the solution process in each case are;

- Locate the intercepts of the curve which are the limits of integration
- Use the definite integral function to find the area (and the volume), the graphic calculator case students were not allowed to use the calculator integral command.
- The final section of the CAS active question requires students to use the solve command on the calculator to find *a* and to eliminate the incorrect response.

	Finding intersection points	Finding the Area	Finding the Volume
1994 Question	$f(x)=\frac{1}{2}x^2 \quad g(x)=2x$ $\frac{1}{2}x^2=2x$ $x(x-4)=0 \;\therefore x=0, x=4$	$\int_0^4 2x-\frac{1}{2}x^2\,dx=5\,{}^1\!/_3$	$\pi\int_0^4\left(2x-\frac{1}{2}x^2\right)^2 dx=\frac{128}{15}\pi$
2000 GC Question	*GC used to find the zeros*  *Decreasing (0, 1]* *Increasing [1, ∞)*	*Area:* *The question required the students to find the area using the definite integral, that is pencil and paper techniques.* $\int_0^4 x-2\sqrt{x}\;dx=-{}^8\!/_3$	*Volume:* *Similarly the definite integral had to be used for the volume calculation.* $\pi\int_0^4\left(x-2\sqrt{x}\right)^2 dx={}^{32\pi}\!/_{15}$

CAS 2000 Question	Finding intercepts	Finding the Area	Finding the Volume	Value for a $V_a=\pi A_a$
	solve(X−A×√X=0,X) X=0 X=A^2	∫(X−A√X,X,0,A²) $\frac{A^4}{2}-\frac{2A\cdot(\lvert A\rvert)^3}{3}$	π∫((X−A√X)²,X,0,A²) $\pi\left(\frac{5A^6}{6}-\frac{4A\cdot(\lvert A\rvert)^5}{5}\right)$	solve(π(A^4/2−2×A×(Ab A=0 A=$\frac{-\sqrt{35}}{7}$

Figure 4 Possible solution strategies for the Integral Calculus questions in 1994 and 2000

The significant difference for the CAS question is the interpretation of the mathematical language of the question that is required. Also the incorporation of the function with a variable in it has resulted in a question with a much greater use of technical language and as a consequence will make it more difficult to respond to for some students.

Differential Equations

Danish Ministry of Education (1994)

> **Opgave 4** *The figure (not shown here) shows a model for the yearly relative growth in the Danish population of skarv (a type of bird) since 1980. According to the model the relative growth is described by* $\frac{1}{y}\frac{dy}{dt}=0{,}34-0{,}013t$
>
> *where y is the skarv population and t is the number of years since 1980. In 1992 it was estimated that there was 156 000 skarver in Denmark. Find the solution f(t) to the differential equation for which f(12) = 156 000. In which year will the population of skarven be the biggest according to the model, and how many will that be?*

Danish Ministry of Education (2000a); Danish Ministry of Education (2000b)

> **Opgave 6.** *A function f is a solution to a differential equation and the graph for f goes through the point P (2, e).* $\frac{dy}{dx}=\frac{y}{\ln y}(x+2),\; y>1,$
>
> *Determine an equation for the tangent to the graph for f in the point P.*
> *Determine the equation and the domain for f.*

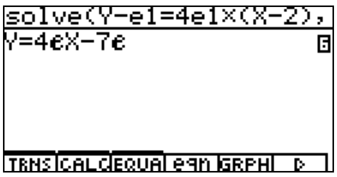
Added part to Opgave 6 (CAS). *Explain why there is no solution to the equation*

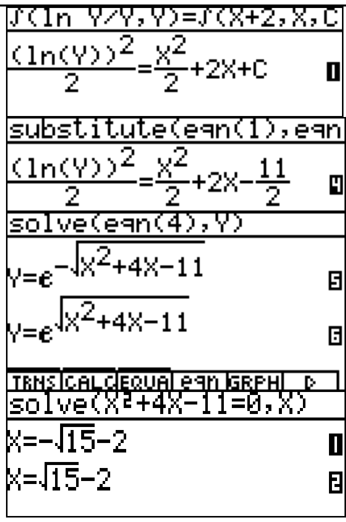
Figure 5 Differential Equation questions from 1994 and 2000

Differential equation questions appeared in each examination and are given in Figure 5. The questions are all first order separable differential equations and students are required to find a particular solution using the given initial values. Interestingly the 1994 question involved the investigation of a real life context whereas the 2000 questions were devoid of context. The CAS question in 2000 required the students to prove why a particular solution did not exist, thus requiring them to select a strategy and then explain why there is no solution, see Figure 6. The latter part of the question represents a shift away from the use of the CAS to find a solution and instead asks them to recognise why there is no solution. We have all too often set questions where there is an expectation of answer, the shift towards proof and explain in this way is to be commended.

	Solution at a given point.	**The equation and the domain for *f*.**
1994	$\frac{1}{y}\frac{dy}{dt} = 0.34-0.013t \ \therefore \ \frac{1}{y}dy=(0.34-0.013t\ )\,dt$ $\ln y = 0.34t - \frac{0.013t^2}{2} + c$ *for* $(12,\ 156\,000)$ $c=8.814$ *and* $y=e^{\left(0.34t-0.0065t^2+8.814\right)}$	Find the when the maximum of the population occurs. $\frac{dy}{dt}=0$ $0=0.34t-0.013$ $\therefore t=26.1538 \qquad and\ y=573\,644$
2000 GC Question	$\frac{dy}{dx}=\frac{y}{\ln y}(x+2), y>1,$ for $(2,e)\frac{dy}{dx}=\frac{e}{\ln e}(2+2)=4e$ Equation of Tangent at $(2,e)$ $y-y_1=\frac{dy}{dx}(x-x_1)$ $y-e=4e(x-2)$ $\therefore y=e(4x-7)$	*Equation of f* $\frac{dy}{dx} = \frac{y}{\ln y}(x+2)$; seperating variables $\frac{\ln y}{y}dy=(x+2)dx$ $\int\frac{\ln y}{y}dy=\int x+2\,dx = \left(\frac{\ln y}{2}\right)^2=\frac{x^2}{2}+2x+c$ for $(2,e)$ $\left(\frac{\ln e}{2}\right)^2=\frac{2^2}{2}+4+c \quad \therefore c=-5.5$ $\therefore\ \ln y=\sqrt{x^2+4x-11}$ and $y=e^{\sqrt{x^2+4x-11}}$ Domain : $x^2+4x-11\geq 0$ Solve quadratic : Domain $=R\setminus[-5.87, 1.87]$

2000 CAS Question

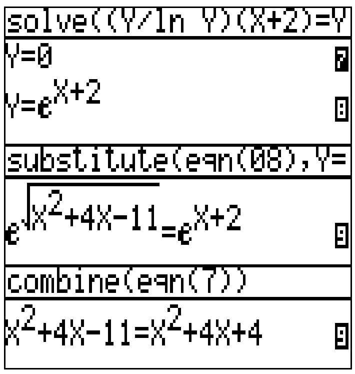




The solution process requires the use of integration, substitution, and solve functions of the CAS.

Explain why there is no solution to the equation $f(x)=f'(x)$.

i.e. $e^{\sqrt{x^2+4x+11}} = \frac{y}{\ln y}(x+2)$



The equations need to be written in a form that enables the student to explain why they are not equal.

Figure 6 A comparison of possible solution strategies for differential equation questions in 1994 and 2000.

The comparison between the 1994 question and the 2000 questions is significant. The level of difficulty in the 2000 graphic calculator question, has increased markedly with the second part requiring considerable algebraic manipulation. This may be a result of the introduction of the graphic calculator and the consequential need to ensure that students cannot use it to complete the question. The steps required to complete the question are similar for each question and include;

- Separating the variables and solving for the given point
- Find the domain either by algebraic methods or using the solve facility for the CAS.
- Recognising that there are limitations in both the use of a graphic calculator and the CAS. Students need to be aware of these difficulties and thus need to have an understanding of the functions involved.

It is evident here that the CAS calculator assists in the solution process by completing the algebraic manipulations for the student but choosing the steps to be taken must be done by the student. That is the focus is on higher-level mathematical skills while the lower level tasks of manipulation are left for the CAS. Or as Kissane has so eloquently stated “It is worth remembering that computer algebra systems on calculators, like those on computers, can handle only symbolic manipulation, which is the least important part of the learning involved.” (Kissane, 2000)

Brand-neutral assessment

From the analysis completed on the examination papers, it is evident that different brand CAS calculators will provide different representations for the same question and in some cases questions can be brand specific. That is, Opgave 7a Danish Ministry of Education (2000b)could be easily completed using one proprietary brand but was less easily completed using an alternative brand, see Figure 7.

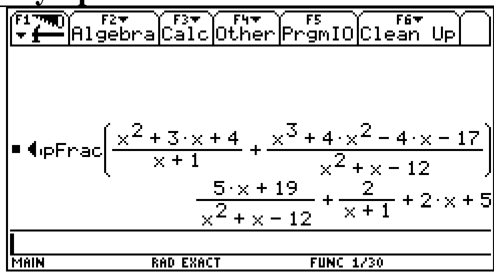
Using Casio Algebra FX2.0 to find the oblique asymptote.	**Using a Texas Instruments TI92 to find the equation of the oblique asymptote.**
There is no direct function on the calculator which will allow you to find the oblique asymptote. Instead the students will need to use the following method. Given that $f(x) = p(x)/h(x)$ Then $f(x) = q(x) + r(x)/h(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. Though the CAS calculator can be used to complete the manipulation, the students will require considerable mathematical insight to find the quotient.	

Figure 7. A comparison of the solution to Opgave 7a using a TI92 and a Casio FX2.0

This raises issues of equity and the ability to write brand neutral examinations questions and further supports the work of Stacey, McRae, Chick, Asp, & Leigh-Lancaster (2000) and McRae & Flynn (2001) who have also raised concerns about the different functionalities of CAS calculators and their affect on the difficulty of questions.

Conclusion

After considering a small selection of problems it appears that the change to examination questions when a CAS is to be incorporated may not be as significant as previously thought. McRae & Flynn, (2001) have also reported similar findings in their investigation of VCE Mathematical Methods Papers where they established that 54% of questions would not need to change to accommodate the use of a CAS. However, in the case of the Danish A level examinations, there is also an examination paper to be completed without any aids plus an oral examination. The impact of these components on the overall assessment scheme should not be underestimated particularly the examination paper which is to be completed without a CAS or books or formulas, this allows for skills to be assessed which are otherwise made redundant by the CAS.

There are some interesting comparison to be made which can provide guidance on the issues related to the implementation of CAS into assessment.

- There appears to be minimal change in questions in questions involving real life exponential function questions. The CAS has a minimal impact on the mathematical understanding that is required to solve the problem.
- By introducing extra variable into equations such as in Integral calculus questions, the language of the question can be made more difficult and possibly as a result less accessible to students.
- Differential equation questions seem less affected by the use of a CAS though there is a reduction in the need for algebraic manipulation.
- Students do need to know the functionality of their CAS and need to be aware of the steps required complete complex problems. The solution method could be different for a CAS than for traditional pencil and paper techniques.

The use of a CAS when completing the questions is only part of the issue; it is evident that students need to have considerable mathematical understanding if they are to use the CAS in an efficient manner. Many authors, including Mitchelmore & Cavanagh (2000), have also recognised the need for a technical understanding of the use of a GC by the students and that complementary changes in curriculum to accommodate the new skills may also be required. These comments are equally applicable to the implementation of CAS into the classroom and assessment, in my view. There is a need for more work to be done in this area to gain a better understanding of the curriculum and training needs of the students and the teachers as they prepare for assessments, which include a CAS component.

The challenge though, will continue to be to develop challenging assessments that will allow students to use a CAS while ensuring that the focus of assessment remains on the determination of the students level of mathematical understanding and not their skill with a CAS nor the functionality of the CAS.

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