Calculating paths in a map using a matrix

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Abstract

I have written several practical educational programs based on computer assisted instruction to help high school students understand applied mathematics or abstract mathematical concepts.[1),2),3),4)] In this case, my work deals with finding paths between two points on a map using a matrix

Finding paths between two points is a visual task easily carried out by our brain; however, if we try to compute the result, several questions arise: How are the calculations implemented? What data is needed? This paper shows how to obtain the paths using simple matrix products. In addition to the paths, we can also easily obtain the total number of paths and distances based on the calculations, not on our intuition. I implemented these calculations with a computer assisted instruction program to help high school students understand the underlying theory and matrix operations.

1. Path Matrix

There are eight points on the map shown in Fig. 1, each point connected to at least another point by a single path. These paths are roads or railway lines. In this case, we make the following 8 by 8 square matrix A: If we can go from point i to point j along a single step path, the (i, j)-entry of the matrix is set to a_{ij}. If no single step path exists between these two points, the (i, j)-entry of the matrix is set to 0. For example, we can go from [1] to [3] along a single step on the map, so the (1,3)-entry is a_{13}; but we cannot go from [1] to [2] in a single step, so, the (1,2)-entry is 0. We call this matrix the "Path Matrix" If there are n points on the map, the Path Matrix is n by n. Examining

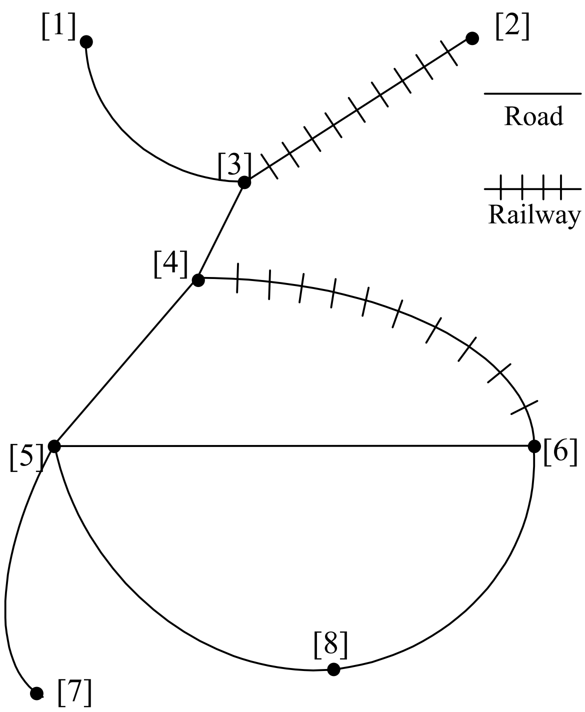

Fig. 1 A map with eight points

the rows of the Path Matrix, we note that the third row has three entries not equal to 0 : a_{31},a_{32} and a_{34}. This means that the point [3] is a fork with three branches. The first row, on the other hand, has only one entry (a_{13}). This means that point [1] is one endpoint of the map. We can thus grasp how many branches each point has by looking at its rows. In the matrix, if entry a_{ij} is nonzero, then entry a_{ji} is also nonzero. However, this is not always the case. It may be possible to go from point j to point i but not from point i to point j. That is, a one-way street.

Path Matrix A

$$\begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & a_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 & a_{45} & a_{46} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 & a_{56} & a_{57} & a_{58} \\ 0 & 0 & 0 & a_{64} & a_{65} & 0 & 0 & a_{68} \\ 0 & 0 & 0 & 0 & a_{75} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{85} & a_{86} & 0 & 0 \end{pmatrix}$$

2. Calculating all the paths between points using the Path Matrix

If we calculate the second power of the Path Matrix A, we obtain all the 2-step paths. (See next page.) For example, the second power of the (4,8)-entry is $a_{45}a_{58}+a_{46}a_{68}$. If we interpret $a_{45}a_{58}$ as the path from the forth point to the eighth point by way of the fifth point (that is [4][5][8]), we deduce from the expression that there are two routes from the forth point to the eighth point, the second one being [4][6][8].

Why does this expression show the second step paths? The entries of the second power matrix (b_{ij}) equal $\sum a_{ik}a_{kj}$, which correspond to the sum of all the single step paths between points i and j. The sum excludes non-existent paths since these have been set to 0 in the Path Matrix. In the example matrix above, the following can be said about the second power of the matrix.

It is natural that all the diagonal entries are not 0. For example, the (3,3)-entry is $a_{31}a_{13}+a_{32}a_{23}+a_{34}a_{43}$. This shows the course of going from the point [3] to another point and returning to point [3]. The number of terms in each diagonal entry corresponds to the number of branches of a point. We can also see that the (2,5)-entry is 0. This means that we cannot go from the point [2] to [5] in 2-steps. We can easily verify this by looking at the map.

If we calculate the third power of the Path Matrix, we obtain all the 3-step paths. In the case of the third power of the Path Matrix A, the (3,8)-entry is $a_{34}a_{45}a_{58}+a_{34}a_{46}a_{68}$. If we interpret $a_{34}a_{45}a_{58}$ as the path from the third point to the eighth point by way of the forth and fifth points, that is [3][4][5][8], we deduce from the expression that there are two paths from the third point to the eighth point, the second one being [3][4][6][8]. Similarly, if we calculate the nth power of the Path Matrix, we obtain all the n-step paths.

If we calculate the expression $\mathrm{A}+\mathrm{A}^2+\mathrm{A}^3+\cdots+\mathrm{A}^n$ for a large n, the matrix shows all the n-step and smaller paths. However, many meaningless paths appear in the matrix which are repetitions of the same path. Generally speaking, if the number of points is N, n may take values

Second power of the Path Matrix A

$$
\begin{pmatrix}
a_{13}a_{31} & a_{13}a_{32} & 0 & a_{13}a_{34} & 0 & 0 & 0 & 0 \\
a_{23}a_{31} & a_{23}a_{32} & 0 & a_{23}a_{34} & 0 & 0 & 0 & 0 \\
0 & 0 & \begin{matrix} a_{31}a_{13} \\ +a_{32}a_{23} \\ +a_{34}a_{43} \end{matrix} & 0 & a_{34}a_{45} & a_{34}a_{46} & 0 & 0 \\
a_{43}a_{31} & a_{43}a_{32} & 0 & \begin{matrix} a_{43}a_{34} \\ +a_{45}a_{54} \\ +a_{46}a_{64} \end{matrix} & a_{46}a_{65} & a_{45}a_{56} & a_{45}a_{57} & \begin{matrix} a_{45}a_{58} \\ +a_{46}a_{68} \end{matrix} \\
0 & 0 & a_{54}a_{43} & a_{56}a_{64} & \begin{matrix} a_{54}a_{45} \\ +a_{56}a_{65} \\ +a_{57}a_{75} \\ +a_{58}a_{85} \end{matrix} & \begin{matrix} a_{54}a_{46} \\ +a_{58}a_{86} \end{matrix} & 0 & a_{56}a_{68} \\
0 & 0 & a_{64}a_{43} & a_{65}a_{54} & \begin{matrix} a_{64}a_{45} \\ +a_{68}a_{85} \end{matrix} & \begin{matrix} a_{64}a_{46} \\ +a_{65}a_{56} \\ +a_{68}a_{86} \end{matrix} & a_{65}a_{57} & a_{65}a_{58} \\
0 & 0 & 0 & a_{75}a_{54} & 0 & a_{75}a_{56} & a_{75}a_{57} & a_{75}a_{58} \\
0 & 0 & 0 & \begin{matrix} a_{85}a_{54} \\ +a_{86}a_{64} \end{matrix} & a_{86}a_{65} & a_{85}a_{56} & a_{85}a_{57} & \begin{matrix} a_{85}a_{58} \\ +a_{86}a_{68} \end{matrix}
\end{pmatrix}
$$

up to N-1 since we can go from one point to another point in N-1 steps or less. (See Fig. 2.) If we calculate $A+A^2+A^3+\cdots+A^{N-1}$ and the (*i*, *j*)-entry of this expression is 0, then, there are no paths from point *i* to point *j*.

Points	[1] ⟶	[2] ⟶	[3] ⟶	[4] ⟶	······· ⟶	[*N-1*] ⟶	[*N*]
Number of paths	1	2	3	4	N-2	N-1	

Fig. 2 Maximum number of paths for N points

3. Calculating the total number of paths between points

If we can go from point *i* to point *j* by 1-step, the (*i*, *j*)-entry in the Path Matrix is set to a_{ij}. Otherwise it is set to 0. If we substitute 1 for all the a_{ij}, this matrix consists of 1s and 0s. The result of changing Path Matrix A like this is shown in Matrix B. If we obtain the second power of

Matrix B (Matrix B^2), the (i, j)-entry of B^2 equals the total number of paths from point i to point j. For example, the (4,8)-entry of the A^2 is $a_{45}a_{48}+a_{46}a_{48}$. We can see that the total number of paths from the forth point to the eighth point is 2 from the expression. If we substitute 1 for a_{45}, a_{48}, a_{46} and a_{48} in this expression, then we obtain the number 2.

Similarly, if we calculate the nth power of the matrix B, we obtain the total number of n-step paths between point i and point j. Therefore, the (i, j)-entry of the $B+B^2+B^3+\cdots+B^n$ is the total number of n-steps or smaller paths.

Matrix B

$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&1&1&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&1&0&0 \end{pmatrix}$$

Matrix B^2

$$\begin{pmatrix} 1&1&0&1&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 0&0&3&0&1&1&0&0\\ 1&1&0&3&1&1&1&2\\ 0&0&1&1&4&2&0&1\\ 0&0&1&1&2&3&1&1\\ 0&0&0&1&0&1&1&1\\ 0&0&0&2&1&1&1&2 \end{pmatrix}$$

By the way, suppose the sum of the ith row of the matrix B equals S_i. When S_i does not equal 0, we divide each entry of the ith row by S_i. The same process is repeated for all the rows. Then, we obtain another matrix B' from B. Suppose that there is a point P moving on the path and the probabilities of moving to the next point with a single step is the same. In other words, if the number of branches of point i is m, then the probability of moving to the next point is $\frac{1}{m}$. The (i, j)-entry of B'^n is then the probability of moving from point i to point j in n-steps. Therefore, B' is a Markov's transition matrix.

4. Calculating the means of transportation

In the map of Fig. 1, there are two means of transportation. One is by car or bus and the other is by train. Here, we calculate the means of transportation as well as all the paths between points.

In the Path Matrix A, if we replace a_{ij} with b_{ij} where the path is railway, the matrix is transformed as shown in Matrix C. That is, a_{23} is replaced with b_{23}, a_{32} with b_{32}, a_{46} with b_{46} and a_{64} with b_{64}.

If we calculate the nth power of the matrix C, we obtain the all the n-step paths between points and the means of transportation on the way.

Matrix C

$$\begin{pmatrix} 0&0&a_{13}&0&0&0&0&0\\ 0&0&\boldsymbol{b_{23}}&0&0&0&0&0\\ a_{31}&\boldsymbol{b_{32}}&0&a_{34}&0&0&0&0\\ 0&0&a_{43}&0&a_{45}&\boldsymbol{b_{46}}&0&0\\ 0&0&0&a_{54}&0&a_{56}&a_{57}&a_{58}\\ 0&0&0&\boldsymbol{b_{64}}&a_{65}&0&0&a_{68}\\ 0&0&0&0&a_{75}&0&0&0\\ 0&0&0&0&a_{85}&a_{86}&0&0 \end{pmatrix}$$

For example, if we calculate the forth power of the matrix C, the (2,8)-entry is $b_{23}a_{34}a_{45}a_{58}+ b_{23}a_{34}b_{46}a_{68}$. This means that we can go from [2] to [8] in four steps. One route is from [2] to [3] by train, from [3] to [8] via [4] and [5] by car. The other route is from [2] to [3] by train, from [3] to [4] by car, from [4] to [6] by train and from [6] to [8] by car.

If we want to add one more means of transportation (i.e. a boat), we can use the letter "*c*" in the matrix to represent it. If there are two means of transportation between point *i* and point *j*, the entry of the matrix C is $a_{ij}+b_{ij}$ and so on.

5. Calculating the distance between points

Consider the distances between points like those in the Fig. 3. The numbers written on the map are the distances between adjoining points. In order to calculate the distance, we transform the Path Matrix into matrix D as follows; if we can go from point *i* to point *j* in one step and distance between them is *p* km, the (*i*, *j*)-entry (a_{ij}) is replaced with a^p. If no single step path exists between these two points, the (*i*, *j*)-entry of the matrix D is set to 0.

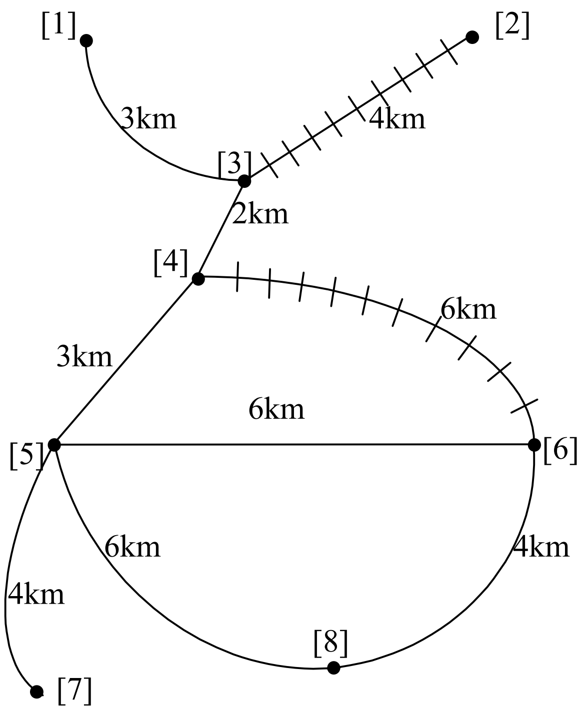

Fig. 3 Distances between points

If we calculate the *n*th power of matrix D, we obtain the distance of all the *n*-step paths by looking at the exponent because if we can go from point *j* to point *k* in one step and the distance between them is *q* km in addition to the assumption above, the (*i*, *k*)-entry of matrix D^2 equals $a^p a^q$ or a^{p+q}. The exponent of "*a*" indicates the distance between point *i* and point *k*.

Matrix D

$$\begin{pmatrix} 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^4 & 0 & 0 & 0 & 0 & 0 \\ a^3 & a^4 & 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 0 & a^3 & a^6 & 0 & 0 \\ 0 & 0 & 0 & a^3 & 0 & a^6 & a^4 & a^6 \\ 0 & 0 & 0 & a^6 & a^6 & 0 & 0 & a^4 \\ 0 & 0 & 0 & 0 & a^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^6 & a^4 & 0 & 0 \end{pmatrix}$$

For example, the (3,8)-entry of A^3 is $a_{34}a_{45}a_{58}+ a_{34}a_{46}a_{68}$.
Also the (3,8)-entry of D^3 is $a^2a^3a^6+a^2a^6a^4=a^{11}+a^{12}$. We can see that there are two ways from [3] to [8] and the distances are 11 km and 12 km respectively.

If we want to calculate the distance of every means of transportation, replace the letter "*a*" with the letter "b" where the means of transportation is by train. Then, the (3,8)-entry of the D^3 is $a^2a^3a^6+a^2b^6a^4 = a^{11}+a^6b^6$. So, we can see that we go by car 6 km and by train 6 km in the course of [3][4][6][8].

6. Connection of the paths

There are two groups of paths shown in Fig.4 (P and Q). Suppose that [1],[2],[3],[4],[5] belong to P and [5],[6],[7],[8] belong to Q. We make the 8 by 8 Path Matrices (A_P and A_Q) for the groups. A_P is made assuming that there are no paths in the group Q, that is a_{56}, a_{65}, a_{67}, a_{76}, a_{68} and a_{86} are 0. A_Q is made assuming that there are no paths in group P, that is a_{12}, a_{21}, a_{14}, a_{41}, a_{34}, a_{43}, a_{45} and a_{54} are 0. If we calculate A_PA_Q, the only nonzero entry is (4,6). If we calculate A_QA_P, the only nonzero entry is (6,4). Therefore, we conclude the following propositions.

1) If A_PA_Q or A_QA_P is not 0, the group of Paths P and Q are connected. The entries of A_PA_Q or A_QA_P which are not 0 shows the paths which connect the groups P and Q.
2) If A_PA_Q and A_QA_P are both 0, P and Q are not connected.
3) If P and Q are not connected by paths, A_PA_Q and A_QA_P are both 0.

Thus we can analyze the topological properties by calculating the Path Matrix.
Moreover, we can conclude the following proposition.

4) Suppose that P_1, P_2, P_3,$\cdots$and P_n are a group of paths P and each P_i is a connected component of P. That is $P=P_1 \cup P_2 \cup P_3 \cup\cdots\cup P_n$ and P_i and P_j are not connected. (See Fig. 5.) If we make each matrix A_{Pi} on the assumption that there are no paths except for P_i, the following expression is deduced.

$$A_P=A_{P1}+A_{P2}+A_{P3}+\cdots+A_{Pn}$$

$$A_P{}^m=A_{P1}{}^m+A_{P2}{}^m+A_{P3}{}^m+\cdots+A_{Pn}{}^m$$

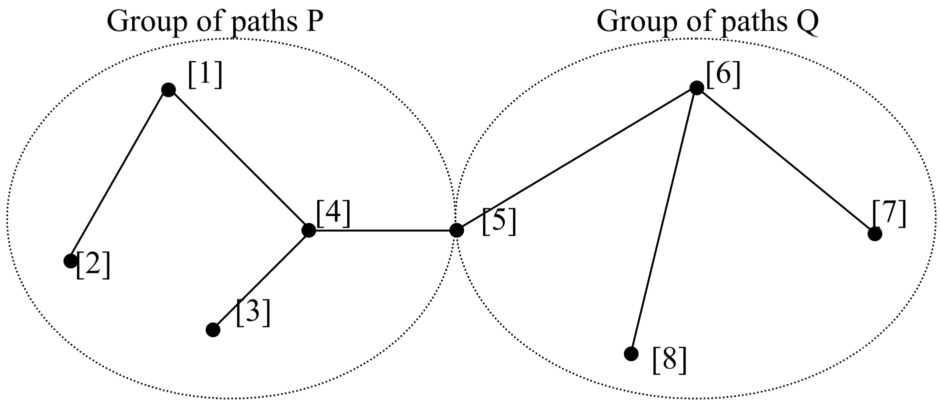

Fig. 4 Two Groups paths P and Q

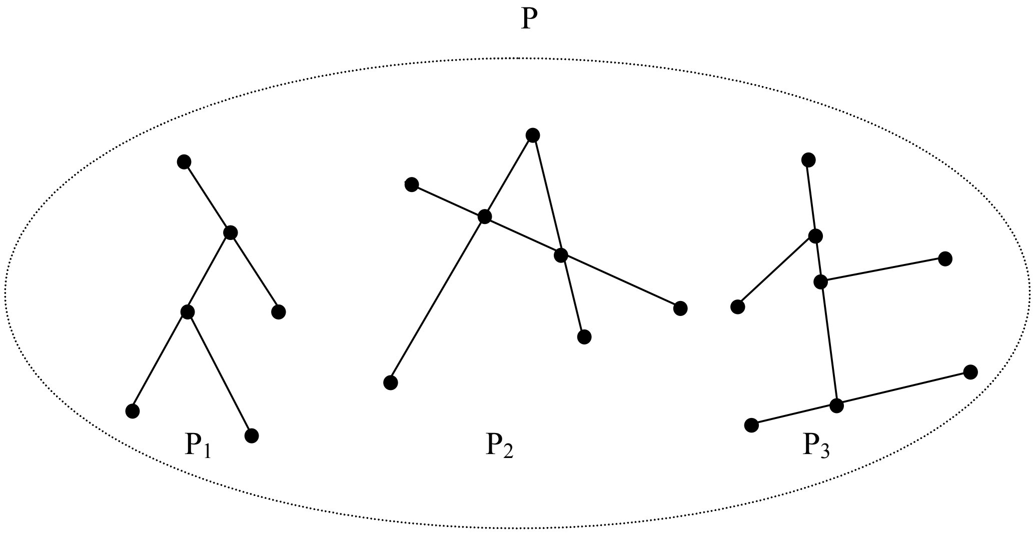

Fig. 5 P_1, P_2 and P_3 are connected components of P

Proof of proposition 1:

Suppose that the (i, j)-entry for $A_P A_Q$ is not 0 and that one of the term of this entry is $a_{ik}a_{kj}$. a_{ik} is the entry for A_P and a_{kj} is the entry for A_Q. Then, we can see that there is a path between point i and k and the path belongs to P. We also see that there is a path between point k and j and that the path belongs to Q. Thus, we can conclude that P and Q are connected.

Proofs for the other propositions are also easily obtained and have been omitted.

7. PATH : a program for computing paths

I wrote the Visual Basic program named "PATH" to calculate the paths, total number of paths between points, means of transportation and distance between points. The main input window is shown in Fig 6. In order to calculate the above items, we must input the several items for the (i, j)-entry on the screen. If no single step path exists between these two points, no input required for the (i, j)-entry. Otherwise, the (i, j)-entry is represented by "a_{ij}", "b_{ij}" or "c_{ij}" In addition, we must input the distance between point i and point j. As can be seen in Fig. 6, the diagonal elements of the matrix do not accept data input.

All the (i, j)-entries are initially set to 0. If we want to input a_{ij} for the (i, j)-entry, the check button corresponding to "a" is clicked. For the distance between point i and point j, the distance is input in the text box. Fig. 6 shows an example of the input.

To calculate the paths, the button labeled "Calculation" is clicked, which changes the screen. Fig. 7 is the screen after clicking this button. (See the page after next.) It includes some buttons to control the rows or columns and the exponents of A, B, C and D. A is the Path Matrix, B consists of ones and zeros for calculating the total number of paths, C consists of a_{ij}, b_{ij}, c_{ij} and zeros for calculating the means of transportation and D consists of a^p and zeros for calculating the distances. In the lower part of Fig. 7, (3,8)-entries of the matrices A^3, B^3, C^3 and D^3 are displayed. We can

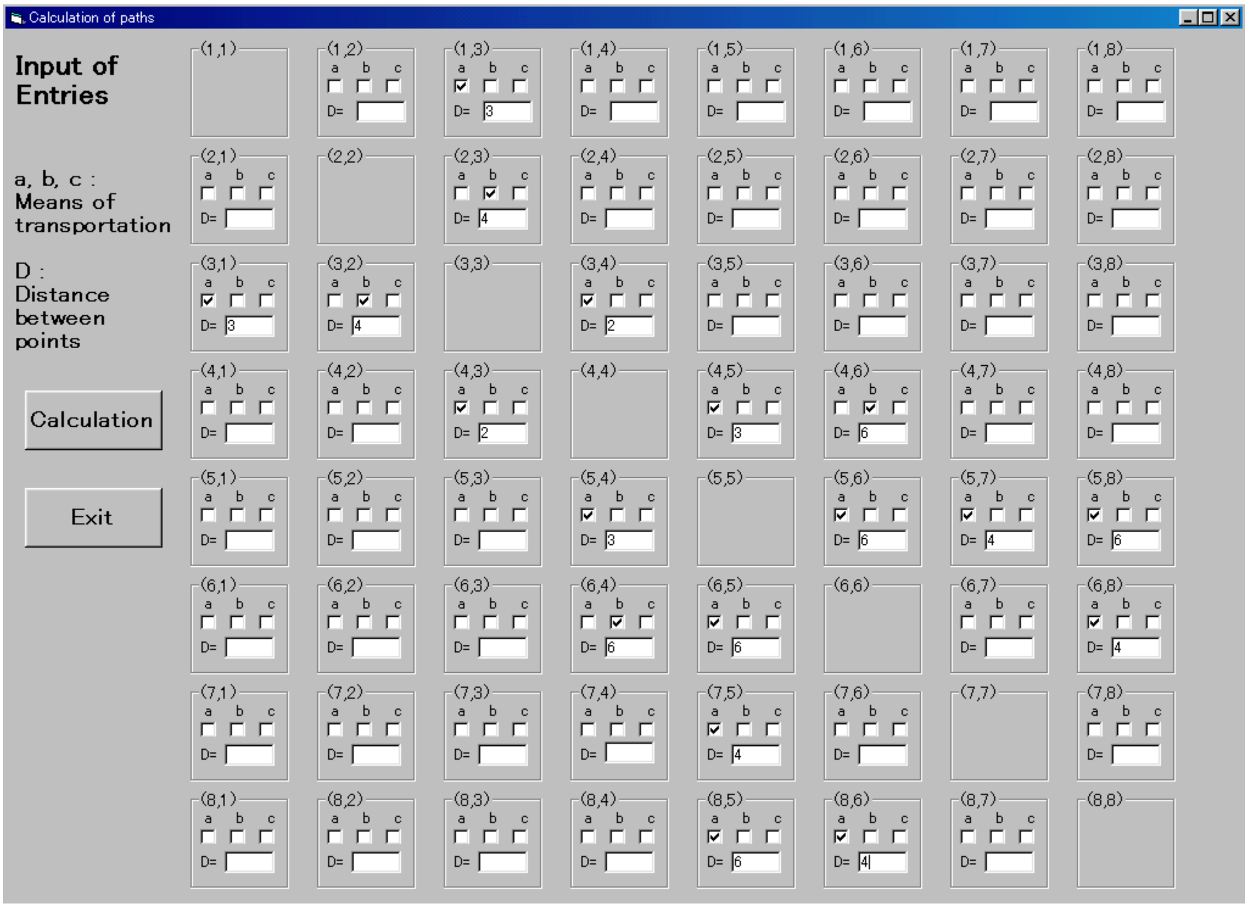

Fig. 6 Main window of PATH

see that there are two courses from [3] to [8] and that one is 11 km and the other is 12 km. To view other entries, the buttons labeled "Row +", "Row -", "Column +", "Column -" are clicked to change the row or column.

If the button labeled "Exponent +" is clicked once, the exponents for A, B, C and D increase by one. That is, in this case, the entries of the forth power of matrices A, B, C and D are displayed. If the button labeled "Exponent -" is clicked once, the exponents of A, B, C and D decrease one. Thus we can calculate the paths, total number of paths between points, the means of transportation and distance between points.

8. Discussion

I taught 12^{th}-grade students how to calculate the paths between points using the program PATH. They showed interest in the matrix model as well as in the way the computer displays the paths between points. Following is a summary of their impressions.

· It is a good mathematical application for the calculating the paths. It was a wonderful experience for me.

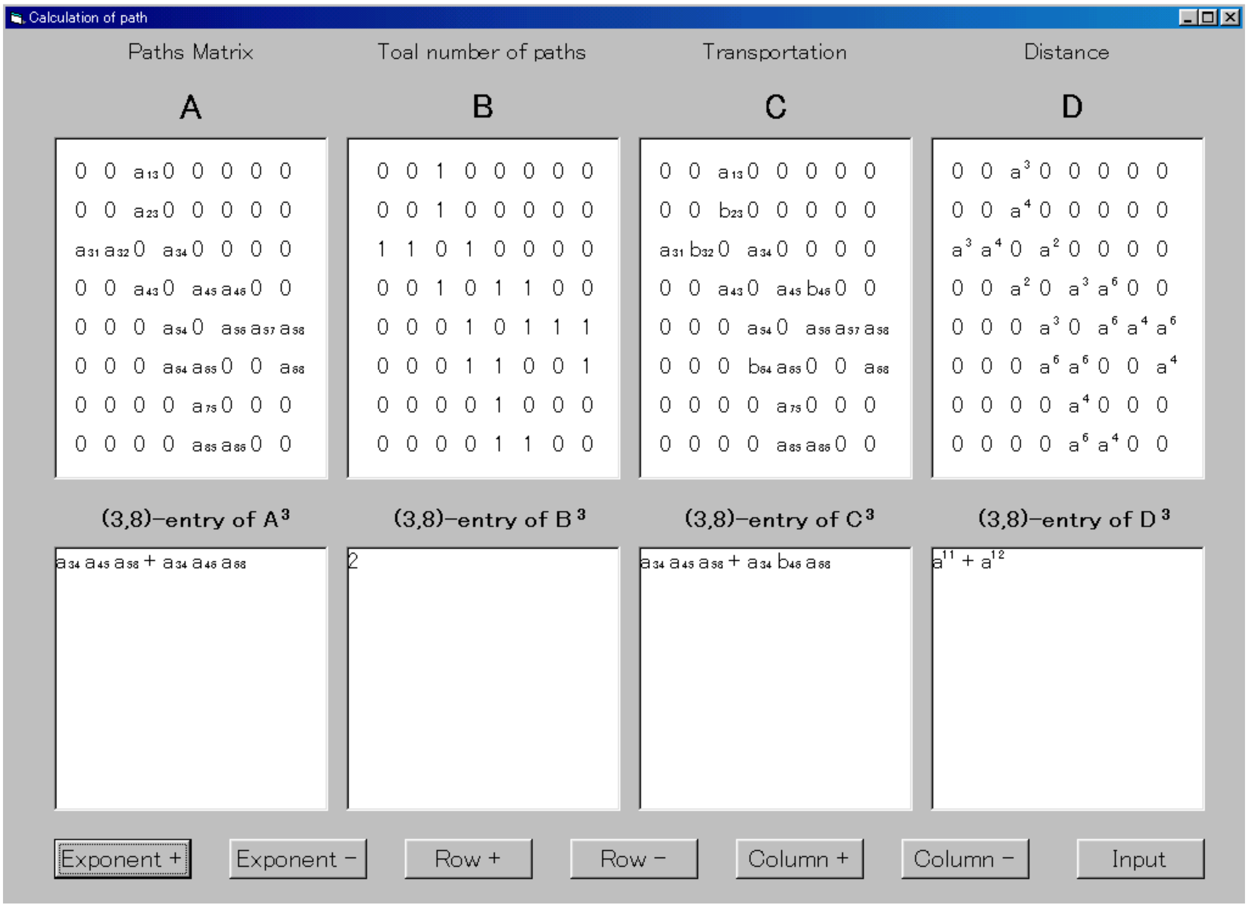

Fig. 7　Result of calculation

- When I first studied matrices, I thought they were uninteresting and useless.　But this model taught me that matrices are important and have practical applications.
- I was surprised that the nth power of the Path Matrix shows the paths between points.　It is a wonderful idea.
- I learned mathematics in order to solve the entrance examination test and considered mathematical knowledge only as a skill for solving problems.　But this lesson has changed my way of thinking.　Mathematics itself is interesting and accessible.
- It was an interesting lesson for me because we can obtain the paths between points based on the calculation not on our intuition.
- This program is good and if you sell your program, you will get the great profit.　It was a fruitful time!
- I am interested in computer method.　I would like to develop a program based on this idea in the near future.
- Matrix product was dull for me.　But now, it has become meaningful thanks to this lesson.　I want to learn more mathematical application.

In Japan, high school students learn about matrices when they take "Mathematics C", but linear mapping is not included. Matrix applications in the high school curriculum include only the solution of simultaneous equation. It is natural, then, that many students consider matrices to be boring and useless. They especially feel doubtful about the product of matrices.

As we can see from the students' impressions, this model makes the students gain interest and helps them understand the meaning about the product. Professor Miwa illustrated the modeling process as shown in Fig. 8 [5] and wrote that mathematical modeling is important in mathematics education since the modeling makes school mathematics applicable.[6] In this case, making Path Matrix, product of the matrices and finding paths between points are the modeling process. This model connects real events with mathematical events, expanding their mathematical world.

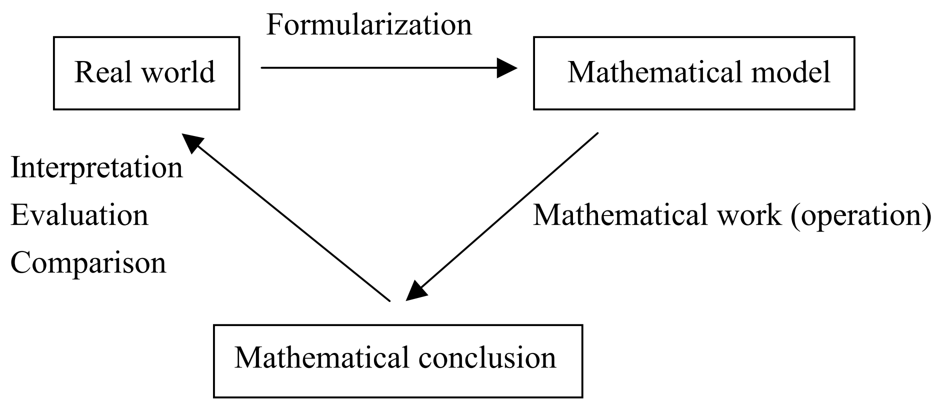

Fig. 8 Modeling process

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