ITERATIONS OF WARING POLYNOMIALS AND CONVERGENCE ACCELERATION

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Abstract. In this paper we introduce two kinds of k-Waring polynomials. Correspondingly, we introduce two kinds of iterated sequences, the k-Waring sequences. The iteration properties of the k-Waring polynomials are discussed. The applications of the two kinds of k-Waring sequences to accelerating convergence are recommended. Fast algorithms for high accuracy computation of square-root and other quadratic irrational number are stated.

Keywords. iteration of function, Waring polynomials, Waring sequences, convergence acceleration, F-L (Fibonacci-Lucas) sequence.

1. DEFINITIONS AND INTRODUCTION

It is known that the Waring formula [6] is

$$\sum_{i=0}^{[k/2]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}(x+y)^{k-2i}(xy)^i = x^k+y^k \ (k\in\mathbb{Z}^+). \tag{1.1}$$

We introduce two kinds of polynomials associated with the Waring formula. They are the k-Waring polynomial of the first kind

$$p_k(x,y)=\sum_{i=0}^{[k/2]}(-1)^i\frac{k}{k-i}\binom{k-i}{i}x^{k-2i}y^i \ (k\in\mathbb{Z}^+), \tag{1.2}$$

and the k-Waring polynomial of the second kind

$$q_k(x,y)=\sum_{i=0}^{[k/2]}\frac{k}{k-i}\binom{k-i}{i}\Delta^{\frac{k-1}{2}-i}x^{k-2i}y^i \ (k\in\mathbb{Z}^+, k \text{ is odd}), \tag{1.3}$$

where $\Delta=a^2+4b\neq 0$.

Correspondingly, we introduce two kinds of iterated sequences. They are the k-Waring sequence of the first kind $\{w_n(k;a,b)\}$ and the k-Waring sequence of the second kind

$\{h_n(k;a,b)\}$ which are defined as

$$w_{n+1}(k;a,b) = p_k(w_n(k;a,b),(-b)^{k^n}),\ w_0(k;a,b) = a, \tag{1.4}$$

and

$$h_{n+1}(k;a,b) = q_k(h_n(k;a,b),(-b)^{k^n}),\ h_0(k;a,b) = 1, \tag{1.5}$$

respectively.

Our main purposes are to deal with the iteration properties of the Waring polynomials and to develop the applications of the Waring sequences. In Section 2 explicit expressions of $w_n(k;a,b)$ and $h_n(k;a,b)$ are given. It will be seen that the expressions are related to F-L (Fibonacci-Lucas) sequences. In section 3 we give a fast algorithm for high accuracy computation of square-root by using Waring sequences. In section 4, for computing other quadratic irrational number, we introduce the Waring transformation. It will be observed that for the computation of quadratic irrational number our method is faster than the Aitken acceleration and other methods.

2. WARING SEQUENCES AND F–L SEQUENCES

Let a and b be constants. In [1] sequence $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ satisfying

$$u_{n+2}(a,b) = au_{n+1}(a,b) + bu_n(a,b),\ u_0(a,b) = 0, u_1(a,b) = 1 \tag{2.1}$$

and

$$v_{n+2}(a,b) = av_{n+1}(a,b) + bv_n(a,b),\ v_0(a,b) = 2, v_1(a,b) = a \tag{2.2}$$

is called generalized Fibonacci sequence and generalized Lucas sequence, of second order, respectively. And in [7] they are said to belong to second order F-L (Fibonacci-Lucas) sequences. Let

$$\Delta = a^2 + 4b,\ \ \alpha = (a + \sqrt{\Delta})/2,\ \ \beta = (a - \sqrt{\Delta})/2. \tag{2.3}$$

Then

$$\alpha + \beta = a,\ \ \alpha\beta = -b,\ \ \alpha - \beta = \sqrt{\Delta}. \tag{2.4}$$

If $\Delta \neq 0$ we have the Binet formulas[1]

$$u_n(a,b) = (\alpha^n - \beta^n)/(\alpha - \beta), \tag{2.5}$$

$$v_n(a,b) = \alpha^n + \beta^n. \tag{2.6}$$

Lemma 2.1.

$$p_k(v_n(a,b), (-b)^n) = v_{kn}(a,b), \tag{2.7}$$

$$q_k(u_n(a,b), (-b)^n) = u_{kn}(a,b)\ . \tag{2.8}$$

Proof. From (1.2), (2.6), (2.4) and (1.1) we have

$$\begin{aligned} p_k(v_n(a,b), (-b)^n) &= \sum_{i=0}^{[k/2]} (-1)^i \frac{k}{k-i} \binom{k-i}{i} v_n^{k-2i} ((-b)^n)^i \\ &= \sum_{i=0}^{[k/2]} (-1)^i \frac{k}{k-i} \binom{k-i}{i} (\alpha^n + \beta^n)^{k-2i} (\alpha^n \beta^n)^i \\ &= (\alpha^n)^k + (\beta^n)^k = v_{kn}(a,b). \end{aligned}$$

From (1.3), (2.5), (2.4) and (1.1), we have

$$\begin{aligned} q_k(u_n(a,b), (-b)^n) &= \sum_{i=0}^{[k/2]} \frac{k}{k-i} \binom{k-i}{i} \Delta^{\frac{k-1}{2}-i} u_n^{k-2i} ((-b)^n)^i \\ &= (\alpha-\beta)^{-1} \sum_{i=0}^{[k/2]} (-1)^i \frac{k}{k-i} \binom{k-i}{i} (\alpha^n - \beta^n)^{k-2i} (\alpha^n(-\beta^n))^i \\ &= (\alpha-\beta)^{-1}((\alpha^n)^k + (-\beta^n)^k) = (\alpha^{kn} - \beta^{kn})/(\alpha-\beta) \text{ (Since } k \text{ is odd)} \\ &= u_{kn}(a,b). \end{aligned}$$

□

Theorem 2.2. *For $n \geq 0$,*

$$w_n(k;a,b) = v_{k^n}(a,b), \tag{2.9}$$

$$h_n(k;a,b) = u_{k^n}(a,b). \tag{2.10}$$

Proof. By the definition we have $w_0 = a = v_1 = v_{k^0}$. Thus (2.9) holds for $n = 0$. Assume that (2.9) holds for n. Then by (1.4), by the induction hypotheses and by (2.7) we get

$$w_{n+1}(k;a,b) = p_k(w_n(k;a,b), (-b)^{k^n}) = p_k(v_{k^n}(a,b), (-b)^{k^n}) = v_{k^{n+1}}(a,b).$$

Hence (2.9) holds for any $n \geq 0$. The proof of (2.10) can be finished in the same way. $\square$

3. FAST ALGORITHM FOR COMPUTING SQUARE-ROOT

In the subsequential discussions of the paper we always assume that $a, b \in \mathbb{Z}$, $ab \neq 0$, and $\Delta > 0$ is not a perfect square. Besides, we always assume that k is odd.

Let d be a positive integer that is not a perfect square. In [5] M. I. Ratliff gave an algorithm which produces a sequence of rational numbers that converges quadratically to the square-root of d. Here we provide an algorithm which produces a sequence of rational numbers that converges by k^{th}-power to the square-root of d for given k.

Theorem 3.1. *Let $d \in \mathbb{Z}^+$ not be a perfect square so that it can be expressed as*

$$d = a_1^2 + b_1 \ (a_1 \geq 1,\ 1 \leq b_1 \leq 2a_1). \tag{3.1}$$

Let

$$t_n = \frac{w_n(k; 2a_1, b_1)}{2h_n(k; 2a_1, b_1)}. \tag{3.2}$$

Then

(1)

$$\lim_{n\to\infty} t_n = \sqrt{d}; \tag{3.3}$$

(2)

$$\lim_{n\to\infty} \frac{|t_{n+1} - \sqrt{d}|}{|t_n - \sqrt{d}|^k} = \frac{1}{(2\sqrt{d})^{k-1}}; \tag{3.4}$$

(3)

$$t_n < \sqrt{d} < t_n + \varepsilon, \tag{3.5}$$

where

$$\varepsilon = \frac{2(16a_1^6 + b_1(8a_1^4 - 2a_1^2 b_1 + b_1^2))(b(8a_1^4 - 2a_1^2 b_1 + b_1^2)^2)^{k^n}}{(16a_1^5)^{2k^n+1}}. \tag{3.6}$$

Remark. (3.3) and (3.4) indicate that the sequence $\{t_n\}$ converges by k^{th}-power to $\sqrt{d}$ for given k; (3.5) estimates the error of t_n as the asymptotic value of $\sqrt{d}$.

Proof. Let $a = 2a_1$ and $b = b_1$. Then from (2.3) we have $\alpha = a_1 + \sqrt{a_1^2 + b_1} = a_1 + \sqrt{d}$ and $\beta = a_1 - \sqrt{a_1^2 + b_1} = a_1 - \sqrt{d}$. Thus $\alpha > 0, \beta < 0$ and $|\alpha| > |\beta|$.

(1) From (3.2), (2.9), (2.10), (2.6) and (2.5) we have

$$t_n = \frac{v_{k^n}}{2u_{k^n}} = \sqrt{d}\frac{\alpha^{k^n}+\beta^{k^n}}{\alpha^{k^n}-\beta^{k^n}} = \sqrt{d}\frac{1+(\beta/\alpha)^{k^n}}{1-(\beta/\alpha)^{k^n}} \to \sqrt{d}\ (n\to\infty);$$

(2) From the above

$$|t_n-\sqrt{d}| = \sqrt{d}\left|\frac{2\beta^{k^n}}{\alpha^{k^n}-\beta^{k^n}}\right|. \tag{3.7}$$

Whence

$$\begin{aligned}\frac{|t_{n+1}-\sqrt{d}|}{|t_n-\sqrt{d}|^k} =& \frac{1}{(\sqrt{d})^{k-1}}\left|\frac{2\beta^{k^{n+1}}}{\alpha^{k^{n+1}}-\beta^{k^{n+1}}}\left(\frac{\alpha^{k^n}-\beta^{k^n}}{2\beta^{k^n}}\right)^k\right| \\ =& \frac{1}{(2\sqrt{d})^{k-1}}\frac{(1-(\beta/\alpha)^{k^n})^k}{1-(\beta/\alpha)^{k^{n+1}}} \to \frac{1}{(2\sqrt{d})^{k-1}};\end{aligned}$$

(3) Expanding $\sqrt{1+x}$ by the Maclaurin's formula we get

$$\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}(1+\theta x)^{-7/2}x^4,\ (0<\theta<1).$$

Then

$$\sqrt{1+x} < 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3. \tag{3.8}$$

Whence for $x>0$

$$\frac{(\sqrt{1+x}-1)^2}{x} < x\left(\frac{x^2-2x+8}{16}\right)^2. \tag{3.9}$$

Since $\beta<0, |\alpha|>|\beta|$ and k is odd, then $t_n<\sqrt{d}$ and from (3.7) we get

$$|t_n-\sqrt{d}| < 2\sqrt{d}\left|\frac{\beta}{\alpha}\right|^{k^n}.$$

Letting $x=b_1/a_1^2$ in (3.9) the last inequality can be transformed to

$$\begin{aligned}|t_n-\sqrt{d}| < 2\sqrt{d}\left(\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right)^{k^n} =& 2\sqrt{d}\left(\frac{(\sqrt{1+x}-1)^2}{x}\right)^{k^n} \\ <& 2\sqrt{d}\frac{(x(x^2-2x+8)^2)^{k^n}}{16^{2k^n}}.\end{aligned} \tag{3.10}$$

Thus the conclusion follows from $\sqrt{d}=a_1\sqrt{1+x}<a_1(16+x(8-2x+x^2))/16$.

□

As an example, we compute the asymptotic value of $\sqrt{19}$. Since $d=19=4^2+3$ we have $a_1=4, b_1=3$ and so $a=8, b=3$ and $\Delta=4d=76$. We take $w_n(3;8,3)/(2\,h_n(3;8,3))$ as the required value. From (1.2) and (1.3) we have

$$p_3(x,y)=x^3-3xy,\quad q_3(x,y)=76x^3+3xy.$$

Whence (1.4) and (1.5) give

$$
\begin{aligned}
w_0 =& 8,\\
w_1 =& 8^3 - 3\cdot 8\cdot (-3)^{3^0} = 584,\\
w_2 =& 584^3 - 3\cdot 584\cdot (-3)^{3^1} = 199\,224\,008,\\
w_3 =& 199\,224\,008^3 - 3\cdot 199\,224\,008\cdot (-3)^{3^2} = 7\,907\,241\,790\,888\,078\,453\,456\,904;\\
h_0 =& 1,\\
h_1 =& 76\cdot 1^3 + 3\cdot 1\cdot (-3)^{3^0} = 67,\\
h_2 =& 76\cdot 67^3 + 3\cdot 67\cdot (-3)^{3^1} = 22\,852\,561,\\
h_3 =& 76\cdot 22\,852\,561^3 + 3\cdot 22\,852\,561\cdot (-3)^{3^2} = 907\,022\,839\,174\,281\,808\,146\,067
\end{aligned}
$$

Thus from (3.2) we obtain

$$
\begin{aligned}
t_3 = w_3/(2h_3) =& w_3(3;8,3)/(2\,h_3(3;8,3))\\
=& 4.\,358\,898\,943\,540\,673\,552\,236\,981\,983\,859\,615\,658\,073\,765\,568\,641\,2
\end{aligned}
$$

Substituting $a_1 = 4, b_1 = 3$ and $k = n = 3$ into (3.6) we obtain

$$\varepsilon < 1.11 \times 10^{-36}.$$

Whence

$$t_n + \varepsilon < 4.\,358\,898\,943\,540\,673\,552\,236\,981\,983\,859\,615\,659\,2.$$

Thus, by (3.5) we get the asymptotic value of $\sqrt{19}$ with 37 significant figures

$$\sqrt{19} \approx 4.\,358\,898\,943\,540\,673\,552\,236\,981\,983\,859\,615\,659.$$

Remark.

(1) We can give $|t_n - \sqrt{d}|$ a rude and simple estimation. Let $f(x) = x^2 - 2x + 8$. Since $0 < x = b_1/a_1^2 \le 2a_1/a_1^2 = 2/a_1 \le 2$, $f(0) = f(2) = 8$ and $f(x) > 0$ we have $f(x)^2 \le 64$ in $(0, 2]$. Then from (3.10) we have

$$|t_n - \sqrt{d}| < 2\sqrt{d}\frac{(64x)^{k^n}}{16^{2k^n}} \le \frac{2\sqrt{d}}{(2a_1)^{k^n}} \tag{3.11}$$

Since $\sqrt{d} < a_1 + 1$ we get

$$|t_n - \sqrt{d}| < \frac{2(a_1+1)}{(2a_1)^{k^n}}. \tag{3.12}$$

(2) Even under the worst condition that $a_1 = 1$, the accuracy of t_n increases rapidly as the number of iterations n increases:

For $a_1 = 1$ and $b_1 = 1$, we get asymptotic values of $\sqrt{2}$ with 20 significant figures for $n = 3$ and with 49 significant figures for $n = 4$, respectively:

$$t_3 = w_3(3;2,1)/(2h_3(3;2,1)) \approx 1.414\,213\,562\,373\,095\,048\,8,$$

$$t_4 = w_4(3;2,1)/(2h_4(3;2,1))$$

$$\approx 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,377.$$

For $a_1 = 1$ and $b_1 = 2$, we get asymptotic values of $\sqrt{3}$ with 15 significant figures for $n = 3$ and with 46 significant figures for $n = 4$, respectively:

$$t_3 = w_3(3;2,2)/(2h_3(3;2,2)) \approx 1.732\,050\,807\,568\,88,$$

$$t_4 = w_4(3;2,2)/(2h_4(3;2,2))$$

$$\approx 1.732\,050\,807\,568\,877\,293\,527\,446\,341\,505\,872\,366\,942\,805\,254.$$

(3) By using the above method we can calculate the quadratic irrational root of the form $(r \pm s\sqrt{d})/t$.

4. WARING TRANSFORMATIOM

Let ϕ be the one, which has a greater absolute, of α and β. It is known that

$$\lim_{n\to\infty} \frac{u_{n+1}(a,b)}{u_n(a,b)} = \phi.$$

To accelerate the convergence, a certain of authors [4], [2], [3] applied the Aitken transformation

$$A(x,x',x'') = (xx'' - x'^{\,2})/(x - 2x' + x'').$$

One of their main results is that for $m \in \mathbb{Z}^+$

$$A^m(r_{n-1}, r_n, r_{n+1}) = r_{2^m n},$$

where $r_n = u_{n+1}(a,b)/u_n(a,b)$.

In this section we present a transformation faster than the Aitken transformation for accelerating the last convergence. In the same way as the last section we can prove that

$$\lim_{n\to\infty} \frac{v_n(a,b)}{u_n(a,b)} = \begin{cases} \sqrt{\Delta} & \text{for } a > 0, \\ -\sqrt{\Delta} & \text{for } a < 0. \end{cases} \tag{4.1}$$

Define sequence $\{X_n\}$ of vectors

$$X_n = (v_n(a,b), u_n(a,b)). \tag{4.2}$$

Define the Waring transformation R_k of $\{X_n\}$ by

$$R_k(X_n) = R_k(v_n(a,b), u_n(a,b)) = (p_k(v_n(a,b), (-b)^n), q_k(u_n(a,b), (-b)^n)). \tag{4.3}$$

Then by (2.7) and (2.8)

Theorem 4.1.

$$R_k(X_n) = (v_{kn}(a,b), u_{kn}(a,b)) = X_{kn}. \tag{4.4}$$

Let s_n be the function of vector X_n:

$$s_n = s(X_n) = s(v_n(a,b), u_n(a,b)) = \frac{a}{2} \pm \frac{v_n(a,b)}{2u_n(a,b)}. \tag{4.5}$$

Define the Waring transformation T_k of $\{s_n\}$ by

$$T_k(s(X_n)) = s(R_k(X_n)). \tag{4.6}$$

By (4.1) and (4.5) it is easy to see that

$$\lim_{n\to\infty} s_n = \begin{cases} \frac{a\pm\sqrt{\Delta}}{2} & \text{for } a > 0, \\ \frac{a\mp\sqrt{\Delta}}{2} & \text{for } a < 0. \end{cases} \tag{4.7}$$

Furthermore, we have the faster acceleration

Theorem 4.2. *For $m \in \mathbb{Z}^+$*

$$T_k^m(s_n) = s_{k^m n}. \tag{4.8}$$

Proof. By (4.5), (4.6) and (4.4) we have

$$T_k(s_n) = T_k(s(X_n)) = s(R_k(X_n)) = s(X_{kn}) = s_{kn},$$

$$T_k^2(s_n) = T_k(T_k(s_n)) = T_k(s_{kn}) = s_{k^2 n}.$$

The proof can be completed by induction. □

As an example, we calculate $\phi = (5+\sqrt{53})/2$, one of the root of the polynomial x^2-5x-7. In this case $a=5, b=7$ and $\Delta=53$. In (4.8) let $k=3, n=4$ and $m=3$. Then

$$p_3(x,y)=x^3-3xy,\ q_3(x,y)=53x^3+3xy.$$

It is easy to obtain $X_4=(v_4(5,7),u_4(5,7))=(1423,195)$. From (4.3) and (4.4) we can get

$$\begin{aligned}
X_{3\cdot 4}=&(v_{3\cdot 4},\ u_{3\cdot 4})=(p_3(v_{3\cdot\ 4},\ (-7)^4),\ q_3(u_{3\cdot 4},\ (-7)^4)=(2\,871\,224\,098,\ 39\,439\,296),\\
X_{3^2\cdot 4}=&(v_{3^2\cdot 4},\ u_{3^2\cdot 4})=(p_3(v_{3\cdot 4},\ (-7)^{12}),\ q_3(u_{3\cdot 4},\ (-7)^{12})\\
=&(23\,670\,164\,102\,419\,511\,976\,900\,320\,098,\ 325\,134\,708\,986\,635\,870\,717\,332\,288),\\
X_{3^3\cdot 4}=&(v_{3^3\cdot 4},\ u_{3^3\cdot 4})=(p_3(v_{3^2\cdot 4},\ (-7)^{36}),\ q_3(u_{3^2\cdot 4},\ (-7)^{36})\\
=&(13\,261\,840\,689\,358\,477\,547\,126\,012\,369\,335\,288\,574\,302\,546\\
&297\,984\,812\,221\,916\,333\,494\,192\,301\,024\,027\,709\,630\,498,\\
&1\,821\,653\,916\,087\,951\,284\,501\,193\,293\,229\,347\,673\,997\,530\\
&816\,709\,149\,896\,760\,276\,510\,649\,401\,359\,934\,202\,208\,640).
\end{aligned}$$

Whence (4.5) implies

$$\begin{aligned}
s_{3^3\cdot 4}=\frac{5}{2}+\frac{v_{3^3\cdot 4}(5,7)}{2u_{3^3\cdot 4}(5,7)}=&6.\,140\,054\,944\,640\,259\,135\,548\,651\,245\,763\,516\,396\,888\,834\\
&841\,288\,238\,719\,189\,090\,895\,642\,057\,869\,312\,453
\end{aligned}$$

which is an asymptotic value of ϕ with 79 significant figures.

Remark. We can also calculate ϕ by using the method in the last section. Since $53=7^2+4$ we have $a_1=7, b=b_1=4$ and so $a=14, \Delta=212$. Taking $k=3, n=3$ in (3.2) we can get

$$\begin{aligned}
\sqrt{53}\approx t_3=&7.\,280\,109\,889\,280\,518\,271\,097\,302\,491\,527\,032\,793\,777\,669\,682\,576\\
&477\,438\,378\,181\,791\,284\,115\,738\,624\,905\,183\,329\,579\,409\,080\,926\,75
\end{aligned}$$

Thus

$$\phi = (5+\sqrt{53})/2 \approx (5+t_3)/2 \approx 6.140\,054\,944\,640\,259\,135\,548\,651\,245\,763\,516\,396\,888\,834$$
$$841\,288\,238\,719\,189\,090\,895\,642\,057\,869\,312\,452\,591\,664\,789\,704\,540\,463\,38$$

which is an asymptotic value of ϕ with 99 significant figures. We observe that the number of significant figures of $s_{3^3 \cdot 4}$ is less than the one of $(5+t_3)/2$ under the same number of iterations 3. To illustrate this we write

$$|s_{k^m n} - \phi| = \frac{1}{2}\left|\frac{v_{k^m n}}{u_{k^m n}} - \sqrt{\Delta}\right| = \sqrt{\Delta}\left|\frac{\beta^{k^m n}}{\alpha^{k^m n} - \beta^{k^m n}}\right|,$$

which is greater than

$$\delta = \sqrt{\Delta}\left|\frac{\beta}{\alpha}\right|^{k^m n}$$

for $\alpha\beta = -b > 0$ or even n, and less than δ for $\alpha\beta = -b < 0$ and odd n. So, in the example we should take an odd $n(\geq 3)$ in stead of $n = 4$. However, for $\alpha\beta = -b > 0$ the accuracy of $s_{k^m n}$ may be lower than the one of $(a+t_n)/2$ under the same number of iterations.

CONCLUDING REMARK

The computer implementation of our method is easy so we omit it.

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