Impact of the graphic calculator on the Calculus curriculum - A study in the polytechnic context.

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Abstract

The accessibility of more advanced tools, such as the graphic calculator, has created a pressing need for mathematics educators to re-examine what is essential learning content. This is an important concern for institutions, such as Temasek Polytechnic, that are lifting the restriction on graphic calculators in examinations. Riding on graphic calculators may mean manual algebraic manipulation becoming less important. How should our Calculus curriculum be revamped to ensure that sufficient fundamental concepts are included? Are there essential changes we should make in teaching Calculus?

1. Introduction

The usefulness of the graphic calculator in teaching Calculus has generated much interest over the past few years. However, attempts to integrate it into curriculum as a learning tool have been conservative. Why this resistance? Most feel that there is a need first to determine what is essential learning content before giving fuller rein to graphic calculators. Ideally, curriculum should ensure that the students gain sufficient mathematical concepts in the course of solving problems. Now, we see the danger of technology fast replacing many conventional learning processes.

For the purpose of reference, we shall base our discussion on the CASIO Algebra FX 2.0 model in this paper. This model contains typical graphical and numerical features as well as computer algebra systems (CAS).

2. Background

In January 2000, Temasek Polytechnic[1] lifted restrictions on calculators for all written examinations[2]. This means that students are free to use any calculator, including graphic calculators with CAS features. With the lowering prices of graphic calculators, we anticipate more students taking advantage of the relaxation.

The presence of graphic calculators will affect curriculum because of the different nature in which a problem is solved and looked at. For the engineering school at polytechnics, this impact will be rather unique owing to the greater emphasis in application in general. We shall look at Calculus as studied by first and second year engineering students at Temasek Polytechnic.

3. Some Objectives

Firstly, we consider some objectives before we decide what should constitute learning content. We recognise that the changes to curriculum should NOT be focused on graphic calculator features [1]. Instead, they should incorporate ways to help students gain a better understanding of concepts introduced using graphic calculator as a tool.

The following would be a reasonable minimum requirement, keeping in mind that the graphic calculator will significantly simplify some algebraic manipulation processes:

(a) Students should have basic algebraic skills to solve simple problems by hand manipulation skills as well as appreciate the techniques used for more difficult problems;
(b) Students using CAS should indicate that they know when to use the appropriate technique for simplifying and evaluating answers;
(c) Students should have reasonable algebraic sense of whether the answer seems correct;
(d) Students should be able to communicate mathematics by effectively showing the logical steps required to come to an answer [2].

In other words, we should allow students to unload some of the heavy algebraic manipulation to the graphic calculator provided they show understanding by demonstrating intermediate steps [3]. As some content in the original curriculum becomes trivialised with the graphic calculator, additional concepts that capitalise on its capability can be introduced. The following are some possible additions to the objectives list:

(e) Students should be able to show graphical and tabular interpretations to the problems they are working on;
(f) Students should appreciate the use of numerical methods on top of analytical methods;
(g) Students should be able to give generalisations and observations to solutions obtained.

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[1] Temasek Polytechnic is one of four polytechnics in Singapore. Graduates obtain a technical diploma after a three-year programme in fields such as engineering, business, design, applied science, and IT.
[2] Besides Temasek Polytechnic, Cambridge 'A' level candidates in Singapore will also be allowed to use graphic calculators for the first time this year (2001) in the subject Further Mathematics comprising Papers 1 and 2. Calculator use, however is restricted to an approved list without the CAS features. This practice is similar to TEE in Australia [4].

In short, the student can be introduced to alternate ways of solving problems besides analytical ones. On the other hand, the task of formulating the problem, connecting known data and evaluating the meaning of solutions is relatively independent of the graphic calculator [5]. These important skills should be retained and enhanced.

4. Learning Content

Engineering students at Temasek Polytechnic take two Calculus courses over two semesters. The first is a basic Calculus course encompassing concepts of limits, differentiation techniques, implicit differentiation, integration techniques, and related problems such as graphs, maximum/minimum problems, related rates and area under the curve. The second is an advanced Calculus course focusing mainly on ordinary differential equations (ODE), such as techniques for solving first and second order ODE's, as well as Laplace and Fourier transforms. There are also topics on sequences and series, including Taylor and Fourier series.

The following menus and functions in CASIO Algebra FX 2.0 are helpful for teaching Calculus concepts in these courses:

(a) Graphic sketching and generating a table related to a function (GRPH-TBL) are the main features of the graphic calculator.
(b) Dynamic Graphs (DYNA) allow dynamic graph sketching by varying the value of the function parameter.
(c) Function memory stores an expression as a function.
(d) Computer Algebra System (CAS) allows symbolic calculations.
(e) Statistical (STAT) contains lists and graphical options that allow graphical sketching based on table of coordinates.
(f) Equation (EQUA) computes the zero(s) of a function easily.
(g) Differential Equation[3] (DIFF EQ) displays the graphical solutions of ordinary differential equations (ODE).

Despite its advanced features, there are still constraints to the Algebra and CAS modes in CASIO Algebra FX 2.0, such as the inability to perform implicit differentiation. On the whole, however, the graphic calculator enables mathematical thinking through a range of symbolic, graphical and numerical visualisations. We will discuss the applications to both courses here.

4.1. Basic Calculus Course

Except for implicit differentiation, virtually all the limit, derivative and integral problems in this course may be solved using the CAS mode of CASIO Algebra FX 2.0. Students will take a much shorter time to work out and master the required techniques with the help of the graphic calculator tool. This can free up some curriculum time to engage in activities previously impossible without graphic calculators. The following are discussions on possible activities:

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[3] The differential equation (DIFF EQ) menu can be downloaded from the Casio website http://www.casio.co.jp/edu_e/resources/add_in/ for free. This is possible since the calculator has the Add-In Software Function.

More exploration for concept building

Students rarely have a good grasp of the limit concept when they first encounter it. Graph sketching and the table of coordinates (GRPH-TBL) provide a platform for visual exploration and observation of function limits. This could be repeated for derivatives and integrals while asking students to form conjectures for formulae using concepts from first principles. For polytechnic engineering students who are not normally challenged to do rigorous mathematical proof, this offers a useful way of validating theory experimentally.

Numerical Methods

It has always been difficult to teach numerical methods in a meaningful way to students due to the limitations placed on hand calculations. This poses a problem as students entering the working world later mostly face problems that are not solved analytically but numerically. With graphic calculators, students can be introduced to numerical methods without too much difficulty such as in the case of integrals. List may be used to compute Riemann sums or trapezoidal sums of different step sizes to obtain approximations for definite integrals. For indefinite integrals, Euler's method with initial condition provide a solution approximation.

Graphical Methods

Many students in our course have very bad "graph sense", which is a pity as graphs do give us plenty of information. This is due in part to the tedious way in which good graphs are hand - sketched. Normally the task seemed so overwhelming that getting the graph out becomes the objective rather than the beginning of investigation. With the graphing ability of the graphic calculator, graphical solutions to problems become attainable. Students can perform analysis of the shape of the graph and identify the features that characterise critical points on the graph. Maximum and minimum problems can be pictured as a whole, not just focussed on a few specific points. Moreover, meaningful analysis can replace straight answers to problems.

More realistic or complex problems

Problems meant to be solved by hand are restricted in scope. Yet engineering students are frustrated when they do not see the application of the skills in a more realistic context. Graphic calculators are good at dealing with less nice looking numbers and more complex equations. This enables more realistic problems to be given without being encumbered by the details. Students can focus on coming up with the mathematical model to the problem and communicating the mathematics clearly.

Interpreting functions theoretically and practically

Many students coming from high school see functions as numbers in place of a relation between two or more variables. Functions presented as graphs and table of coordinates give a different perspective to symbolic representations. It is also easier to interpret function behaviour when they are presented graphically and numerically. A useful means of exploration also comes from the instant visual feedback when students change function parameters, such as coefficients and intercepts.

4.2. Advanced Calculus Course

With the advanced Calculus course, less content is affected by the graphic calculator. Nevertheless, portions of the technique can still be passed on to the tool. This includes technical skills such as partial fraction decomposition, curve sketching and definite/indefinite integration. In place of it the following concepts may be emphasised: mathematical modelling of engineering problems, interpreting mathematical solution to real problems, and concepts of series convergence and divergence in approximating function values. Besides, graphic calculators also provide different approaches to teaching and learning.

Graphic Interpretation of First Order Ordinary Differential Equations (ODE)

First order ordinary differential equations, generally denoted by the form $\frac{dy}{dx} = f(x, y)$, are closely related to the concept of indefinite integration. The graphical feature of the graphic calculator can give quick solutions and demonstrate some useful concepts.

Take the trivial example of the form $\frac{dy}{dx} = f(x)$ with the general solution $y = \int f(x)dx$. Using the differential equation (DIFF EQ) menu, we can obtain the graphical solution of the given ODE instantly, without the usual time-consuming manual computations. The DIFF EQ menu enables students to explore and observe the graphical meaning of the first order ODE and its relation with the indefinite integral of a function. In particular, we may observe that the direction of the graph $y = \int f(x)dx$ at $x = a$ is determined by $\left.\frac{dy}{dx}\right|_{x=a} = f(a)$ (see Figure 1).

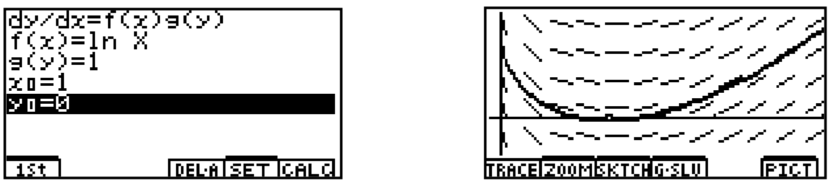
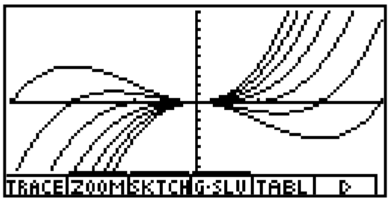
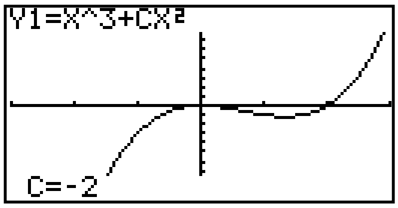


Figure 1. Graphical interpretation of $y = \int \ln(x)dx$

Students often have problems visualising the solution of an ODE as a *family of functions*, not merely *a function*. It is important that students should recognise the former as a general solution and the latter as a particular solution. The DYNA and GRPH-TBL menus are useful in demonstrating the family of solutions for an ODE (see Figure 2a). In addition, these graphs also help students establish a relation between the initial values and the solution of initial value problems (IVP) (see Figure 2b).



2a



2b

Figure 2a. A few members of a solution of an linear ODE $x\dfrac{dy}{dx} - 2y = x^3$;

Figure 2b. A particular solution of an IVP: $\begin{cases} x\dfrac{dy}{dx} - 2y = x^3 \\ y(2) = 0 \end{cases}$

Analytical Methods for solving ODE

Most of the analytical methods used to solve first order ODE's involve various techniques of integration, ranging from very simple to more complicated ones such as integration by parts or the use of partial fractions. Second order homogeneous ODE's with constant coefficients require students to solve auxiliary equations which are quadratic in nature. With the CAS feature of the graphic calculator handling these problems, students can focus on formulating the ODE and analysing the solution of the ODE, both of which are valuable but often under-emphasised skills in the classroom.

Numerical Methods for solving ODE

Two basic methods can be introduced in this course : Euler's method and second order Taylor method. These methods use the iterative process to obtain an approximation of the solution of an ODE with initial conditions. The use of function memory makes the iterative process more practicable.

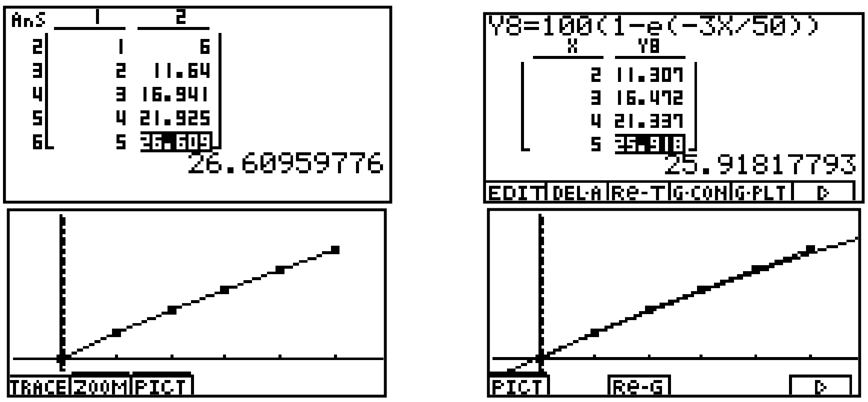
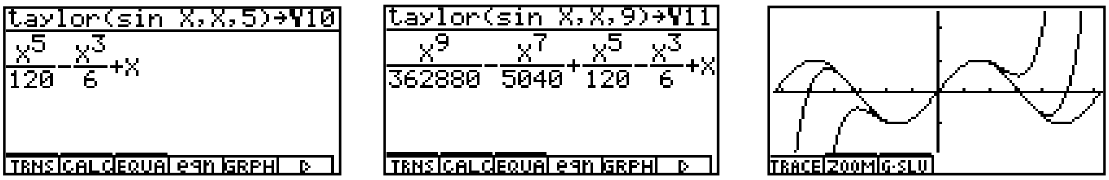


Figure 3. Euler's method vs. analytic solution of an IVP $\begin{cases} \dfrac{dy}{dx} = 6 - \dfrac{3x}{50} \\ y(0) = 0 \end{cases}$

It would be meaningful for students to compare the approximated results with the analytical result graphically (see Figure 3). Plotting the numerical solution of an IVP is made possible with the statistical (STAT) menu, where numerical values are regarded as pairs of statistical data. Though the students are not expected to analyse the error of an approximation, we can expect them to understand the concept of *error* in approximation through comparison and observation. They can observe the accuracy of the numerical solution by varying parameters, such as step-size or number of iterations.

Taylor and Fourier Series

Analytical illustrations of Taylor or Fourier Series to approximate functions can be problematic. Students often fail to see the connection between the given function and its series approximation. By varying the parameters, Taylor Series approximations with different orders can be sketched on the same screen and their subsequent approximations observed (see Figures 4a, b, c).



4a 4b 4c

Figure 4a,b. Taylor Series approximation of order 5 and 9 for sin(x)
Figure 4c. Comparing Taylor Series approximations with original function

Students may also be asked to analyse the characteristic of each type of series approximations by graphical visualizations. It would be good to drive home the point that the Fourier Series is used to approximate a periodic function, while the Taylor Series is used because of its simple polynomial form.

Laplace and Fourier transforms

The strength of the Laplace transform lies in changing a difficult IVP into a manageable algebraic equation by transforming $f(t)$ in t-plane into $F(s)$ in s-plane. Having solved the algebraic equation, the solution of the IVP is obtained by applying the inverse of Laplace transform, L^{-1}, to the algebraic solution.

While Laplace transform is intended to simplify an IVP into an algebraic problem, the problem of simplifying the resulting algebra form is non-trivial. In most cases, we need to find the partial fraction decomposition before we look up the inverse of this algebraic expression from the formulae table for Laplace transforms. The inability to the graphic calculator to do more than evaluating the partial fraction (using CAS on the graphic calculator) makes the technique very cumbersome for students.

It is tempting to remove Laplace transform from the course in view of the numerical methods available on the graphic calculator to solve IVPs. However, the Laplace transform is considerably important despite its restricted use. For one, Laplace transform enables exact solutions to be obtained for IVPs in closed form, and essentially handles linear ODE of any degree. Secondly, the application of Laplace transform significantly simplifies problems in engineering such as circuit analysis. Not to mention, important techniques such as Fourier Transform, z-Transform and wavelets contain concepts that are complicated variations of the simpler Laplace Transform. As such, Laplace transform is likely to remain in the curriculum for some time to come.

5. Assessment Issues

Assessment plays an important role in making the curriculum effective. Content knowledge needs to be assessed as before, and to a greater extent now, process skills (observe, explore, interpret) as well. Examination problems should be set in a way that assesses the learning done in class in these areas. Two types of examination problems would be useful, namely, concise and long problems.

Concise problems assess simple concepts that form the underlying basis of the course. These should be easily evaluated using the graphic calculator, or else, calculator neutral in general. Some examples could be:

Example 1: Given $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & ; x \neq 2 \\ a & ; x = 2 \end{cases}$, determine the value of a so that $f(x)$ is continuous.

Example 2: Give a counter-example to the statement: " Since $f''(2) < 0$, we conclude that there is a local maximum at the point $x = 2$". Give a sufficient condition so that the conclusion is true, illustrating this with a suitable example.

Example 3: Find the area enclosed by the graphs $y = x\sin x$, the x – axis, the y – axis, and $x = 2$.

Example 4: Write the auxiliary equation that solves $y'' + 4y' + ky = 0$. What values of k gives the solution $y = C_1 e^{\alpha x} + C_2 e^{\beta x}$, where α, β are distinct, real numbers?

Example 5: What is the inverse Laplace transform of $\dfrac{1}{(s-2)(s+4)}$?

Longer problems, on the other hand, assess process skills and concepts in a more indirect way. They capitalize on the exploratory strength of the calculator and are modeled after similar activities in class. These could take the form of the examples below:

Example 6: A pipe is carried horizontally around a right-angled bend as shown in Figure 5. The widths of the corridors before and after the bend are a metres and b metres respectively.

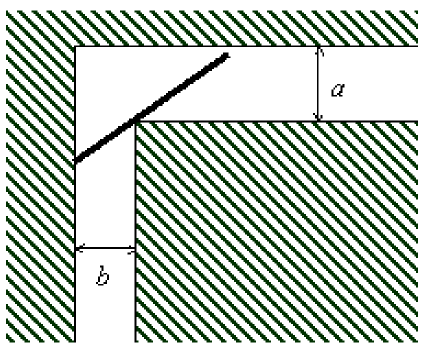


Figure 5

(i) Construct an equation that helps us to find the longest possible pipe that can go round the bend.
(ii) Assuming a = 2.10 metres and b is 1.35 metres, sketch the graph obtained in part (i) and indicate the longest pipe allowed. Describe how you arrived at your answer.
(iii) Supposing b is not fixed yet (the corridor may be widened or narrowed). We want to find the longest pipe allowed for different values of b. Show a table of longest pipe lengths for different values of b, for $1.00 \le b \le 2.50$ in step sizes of 0.05 metres.
(iv) Discuss the effect of b on the longest pipe allowed, illustrating this with suitable graph(s).

Example 7: Given that $\frac{dy}{dx} = \frac{\sin\sqrt{x+2}}{\sqrt{x+2}}$ is defined for $0 \leq x \leq 2\pi$ and $y(0) = 1$,

(i) Write the analytical solution and sketch its graph
(ii) Using Euler's formula, show how you can obtain an approximate solution using 10 iterations. Construct the approximate solution graph.
(iii) Discuss your observations of graphs in parts (i) and (ii). Explain how a more accurate graph can be obtained using numerical methods. Illustrate this with suitable examples.

A different mode of assessment will emerge, perhaps not as structured as before. While concise problems do not differ much from problems set traditionally, some will become easier using the graphic calculator. This necessarily leads to adjustments to mark allocations that stress key concepts rather than algebraic manipulation skills[6]. In contrast to these self-contained problems, long problems should be extensions of work done in the classroom. Taking the form of group projects or individual exploration problems, they build up the students' process skills in problem solving. With sufficient emphasis on classwork, the development of students' process skills will gain suitable recognition in the entire assessment of the student.

6. Some Comments

Bringing in the graphic calculator presents some awkward situations to the teaching of Calculus. It is difficult to design curriculum that will retain most of the mathematical rigour we wish to see, despite the additional exploratory activities put in. It is inevitable that some mathematical concepts will have to suffer, and some portions of the course will become easier for most students. However, this also means that a wider pool of students will have access to these content formerly closed to them due to their weak algebraic skills. This is a practical advantage for engineering students who can apply their knowledge to their work situations rather than not having them at all[4]. For the better students who wish to be challenged with more mathematical rigour, perhaps a separate course that bans the use of graphic calculators could be arranged to cover this area[5] [7].

One of the aims of polytechnic education is to make students ready for the working world [8]. For the engineering school, this means developing skills that are, as far as possible, transferable to the industry that they will enter later. Equipping students with the technology to solve their problems, and building up their problem solving skills through exploration, conjecturing, testing, analysis that are ATTAINABLE will be very important indeed. We propose that the graphic calculator is a good choice of tool for the engineering student.

We do not assume that the graphic calculator will benefit students at all levels of ability. Students with very weak foundations are likely to remain at the bottom and suffer bigger gaps in understanding as larger chunks of mathematics are taken out of the curriculum. Then again, their problem may be better dealt with through remedial work as in any system of learning. Perhaps, of

[4] Many engineering students consider the engineering mathematics modules among the toughest subjects in their course of study, and students at Temasek Polytechnic are no exception.
[5] One example is a third year elective advanced mathematics course that is underway at the engineering school of Temasek Polytechnic. This course is designed for students with an aptitude for mathematics, and who wish to pursue an engineering degree at a local or foreign university later on. Similar courses are run at other local polytechnics.

more importance will be the impact on the student majority with average ability in mathematics. The graphic calculator provides more avenues in understanding concepts and solving problems, which can benefit students with different areas of aptitude. Good students may find the course easier, but they can gain a wider perspective through different approaches to a problem. Properly designed projects can also push them to their own limits of understanding above the others.

7. Conclusion

The graphic calculator is a promising tool that can enable useful shifts in the Calculus curriculum to benefit engineering students in a practical way. However, we need to be aware of the consequences of these changes. It is important that learning objectives are properly set up, then consciously adhered to in the design and execution of the course. Using a versatile tool such as the graphic calculator may also signify a less structured form of learning. In this case, both the participation of students and facilitation of educators will be important in making learning effective. Whether the changes are worthwhile remains to be seen over a longer period of trial.

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