Technological Impact of Modern Abstract Mathematics

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Abstract

Computer technology is requires more advanced mathematics to be used to model quite novel and previously has never experienced world called cyberworlds. As such, this research visits certain nonlinear mathematical domains mostly in algebraic topology. Homotopy, curvature and cellular structured spaces are visited to explore their roles in versatile technological applications. Actually, most of modern high technology manufacturing employs cellular workspaces called work cells. As such the directions we present in this work is technologically quite fundamental and essential.

1 Backgrounds and Motivations

Recent advances of modern abstract mathematics are opening up new doors in technology and applications. Often the human common sense bound by the traditional mathematical framework limits our technological capability.

For example, computer display technology is basically relying on projective geometry that has been serving as the framework of human view representation. Hence, a displayed image is usually a result of the projection of objets as seen from a single viewpoint. Cellular spatial structures allow more generalized computer display technology with multiple and concurrent viewpoints. For automobile assembly, concurrent assembly is common and various parts are assembled from multiple directions simultaneously. There is no common display technology appropriate for such applications. Rapid prototyping is also benefited from cellular spatial structures to realize complex shapes as seen from various directions. Dental care, for example, requires this type of rapid prototyping.

Another example is free form generation. Free forms are widely used in industrial product design. Traditionally free forms have been generated from polygons (e. g. triangles) by modifying their surfaces by splines or some free form functions. They can be directly generated homotopically. Cellular spatial structures with a homotopic framework are a typical example of modern abstract mathematics in algebraic topology where versatile technological applications are abundantly discovered [1], [2].

2 Roles of Invariants in Information Technology

At the crucial time to get into the 21^{st} century in a couple of years, information technology is leading our society to unique cyberworlds in cyberspaces spanning on networked computers [3], such as electronically driven international financial worlds controlling the real worlds constantly. For information technology to go along well with human aesthetics for our daily mental health in high-tension modern life, it has to have artistic provisions as well as technological advanced tools. Information broadcasting as interactive high bandwidth communication is expected to highlight information technology. All information will be basically animated because information reflects dynamically changing worlds on one hand and animation artistically better appeal to people as analogue TV has been on the other hand. As are the physical worlds governed consistently by physical laws build on physical invariant preservation such as mass and energy preservation, for animation to be consistent for validity it has to preserve its own invariants.

There are a couple of invariants in animation. One most fundamental invariant is the homotopic invariant defined through the homotopic equivalence relationship. Another invariant is defined through curvature. For any object moving through translation and rotation to be identified uniquely, the curvature along the outline of the object is a motion invariant.

3 Homotopic Equivalence

The change of cyberworlds in cyberspaces has to be *homotopic*, meaning continuously deforming, for our animation design to maintain the *generality* in terms of homotopic equivalence. We explain this briefly here. Let us consider the changes of a mapping function f relating a cyberspace X to another cyberspace Y. After the change, f becomes another mapping function g. In short, we are designing animation, we are designing the continuous deformation of f into g where f, g: X   Y. We consider the deformation during the normalized interval I = [0, 1] that can be a time interval or a space interval. X and Y are topological spaces. Let us denote the unchanging part A of the cyberspace X as a subspace A   X. Then, what we are designing is a *homotopy* H, where H: X   I   Y such that (x   X) (H (x, 0) = f (x) and H (x, 1) = g (x)), and (a   A, t   I) (H (a, t) = f (a) = g (a)). F is said *homotopic* to g relative to A, and denoted as $f \simeq g$ (rel A). Now here is a new design problem. That is, how we can design two topological spaces X and Y to be *homotopically equivalent* $X \simeq Y$, namely *of the same homotopy type*. It is done by designing f: X   Y and h: Y   X such that $h \circ f \simeq 1_X$ and $f \circ h \simeq 1_Y$, where 1_X and 1_Y are identity maps 1_X : X   X and 1_Y : Y   Y.

Homotopy equivalence is more general than topology equivalence so that homotopy equivalence can identify a changing cyberspace that is topologically no more equivalent after the change. As an example, let us look at the human body. By the way, the human body is so complex that it is often called a micro cosmos, and it itself is a cyberspace. What we are looking at is a very simple case for an illustration purpose only, but is enough for our purpose. Given a silhouette of the human body of a person dancing. Fig. 1 shows how we can identify it automatically while the silhouette shape is changing during the dance. In manufacturing automation, while a component goes through a manufacturing line, for example to get deformed to a desired shape by a group of automated press machines, the deformation process is specified by a homotopy and validated by homotopy equivalence. In medical imagery, from a set of computed tomographical 2D sliced images, we can reconstruct 3D images homotopically in a following way: considering that a slice is homotopically deformed to the next slice, and then from the algorithm derived

Homotopically equivalent and topologically equivalent

Homotopically equivalent but topologically not equivalent

Fig. 1 Homotopy equivalence is more general than homeomorphism as topological equivalence.

from the law of the formation of the organ of which the images are obtained tomographically we can record all the intermediate shapes between the two neighboring slices faithfully without loss [4]. The same applies to the topography case. Further applications are in financial trading areas where from discrete financial data, we can build the entire features automatically and then conduct automated financial trading in a cyberspace based on the expected financial value change shapes thus homotopically derived.

4 Curvature as a Motion Invariant

An efficient method of orientation- and location- independent identification is realized only by finding a small set of orientation- and location- independent

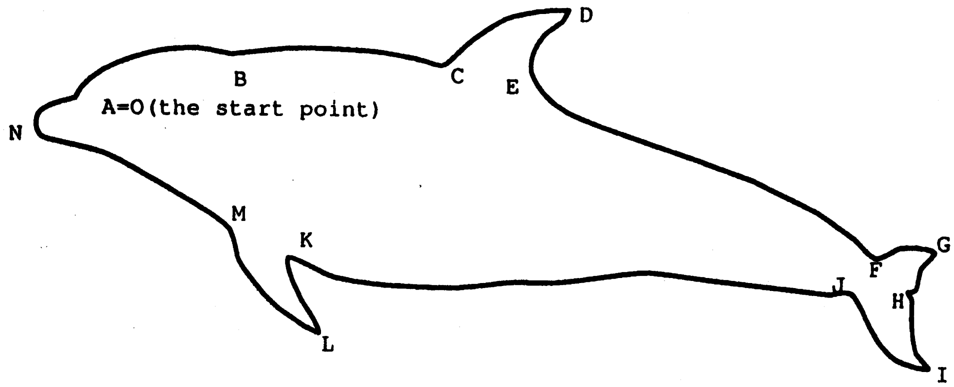


Figure 2: The Outline Curve of a Dolphin Silhouette.

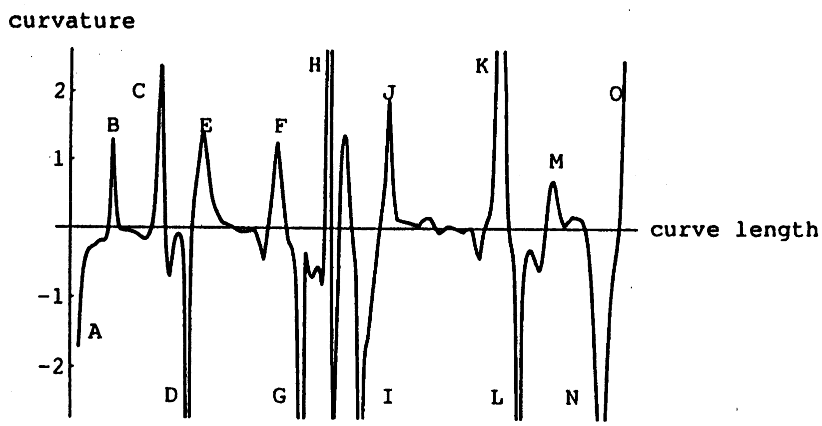


Figure 3: Curvature along the Curve.

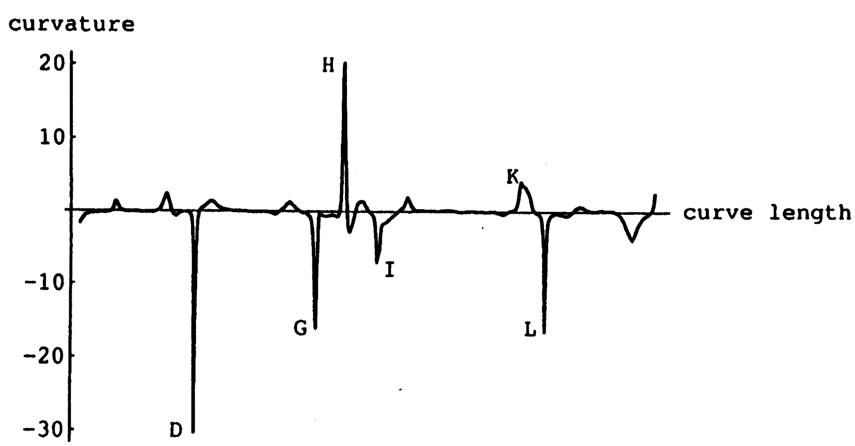


Figure 4: Curvature along the Curve.

invariants of shapes. As such, we can utilize the curvature of shapes as shown

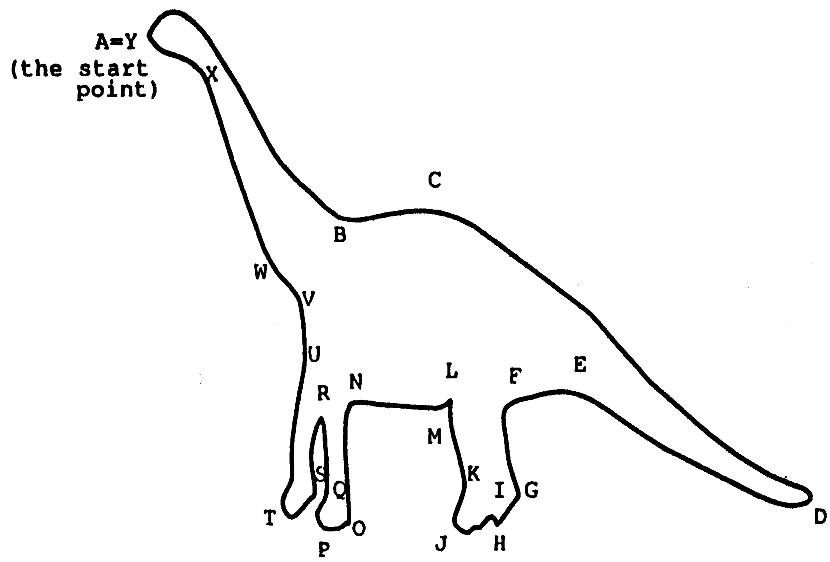


Figure 5: The Outline Curve of a Dinosaur.

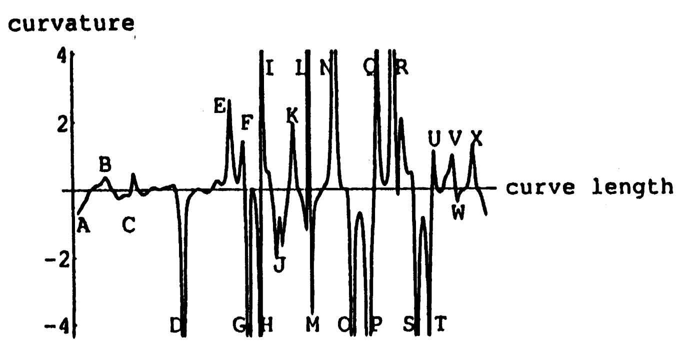


Figure 6: Curvature along the Curve.

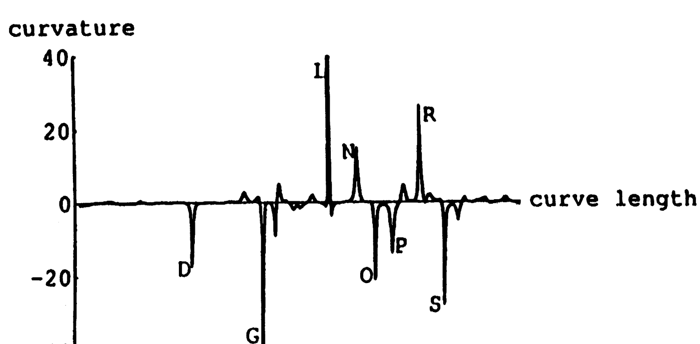


Figure 7: Curvature along the Curve.

previously. Given the curvature of a shape, we can characterize it by characteristic points such as peaks, pits and passes as a small set of orientation- and location- independent invariants. This means a drastic increase in the efficiency of the shape identification. For all the components, we store the sequences of characteristic points. Then we can identify any components flowing through the manufacturing line by comparing small sets of symbols, namely the sequences of characteristic points. Further, we can usually reduce the identification problem of this type to the template-matching problem of circular codes consisting of characteristic point symbols. As a simple example we showed a case of the identification of animals from their silhouettes in Figures 2-7 [5]. The generalization of the examples to higher dimensional cases is under study.

5 Cellular Structured Spaces

There are a number of reasons we need to go beyond traditional liner spaces. Computer display technology, for example, is basically relying on projective geometry that has been serving as the framework of human view representation. Hence, a displayed image is usually a result of the linear projection of objets as seen from a single viewpoint. Here, cellular spatial structures serve to provide more generalized computer display technology with multiple and concurrent viewpoints. As an application, concurrent assembly is common in automobile assembly, and various parts are assembled from multiple directions simultaneously. There is no common display technology appropriate for such applications. Very popular technology includes rapid prototyping, and it is also benefited from cellular spatial structures to realize complex shapes as seen from various directions. Historically dental care has been practicing rapid prototyping and it requires this type of rapid prototyping.

We are able to design cyberspaces as *cellular structured spaces*, in short a *cellular space* [6] as shown in the following. First of all, a *cell* is designed as a topological space X that is topologically equivalent (namely homeomorphic) to an arbitrary dimensional (say n-dimensional) ball B^n, and called an n-cell. From X, we can design a finite or infinite sequence X_p of cells that are subspaces of X, indexed by integer ∧, namely { X_p : p ∧ } called a *filtration*, such that

X_p covers X,

namely $X = {}_{p\ \wedge} X_p$, and

X_{p-1} is a subspace of X_p,

namely $X_0 \quad X_1 \quad X_2 \qquad X_{p-1} \quad X_p \qquad X$.

We can design, thus, a quite general cellular space called a *filtration space* for a cellular space X, as a space X with a filtration designed above, and denote it by { X; X_p : p ∧ }. We can actually build a little bit more structured cellular space, and hence not as general as a filtration space. It is called a closure finite and weak topology space, abbreviated as a *CW – space* constructed by the closed subspaces X_p of X. It is enough for our design with finite numbers of cells. If we need to think about an infinite case, some extra care is required. Further, as in the most cases in natural sciences as seen in theoretical physics, smoothness, namely the existence of continuous derivatives of all orders, is assumed, and sometimes *diffeomorphism,* namely differentiability with a differential inverse is further assumed to turn a CW-complex into a more special case named a *manifold*.

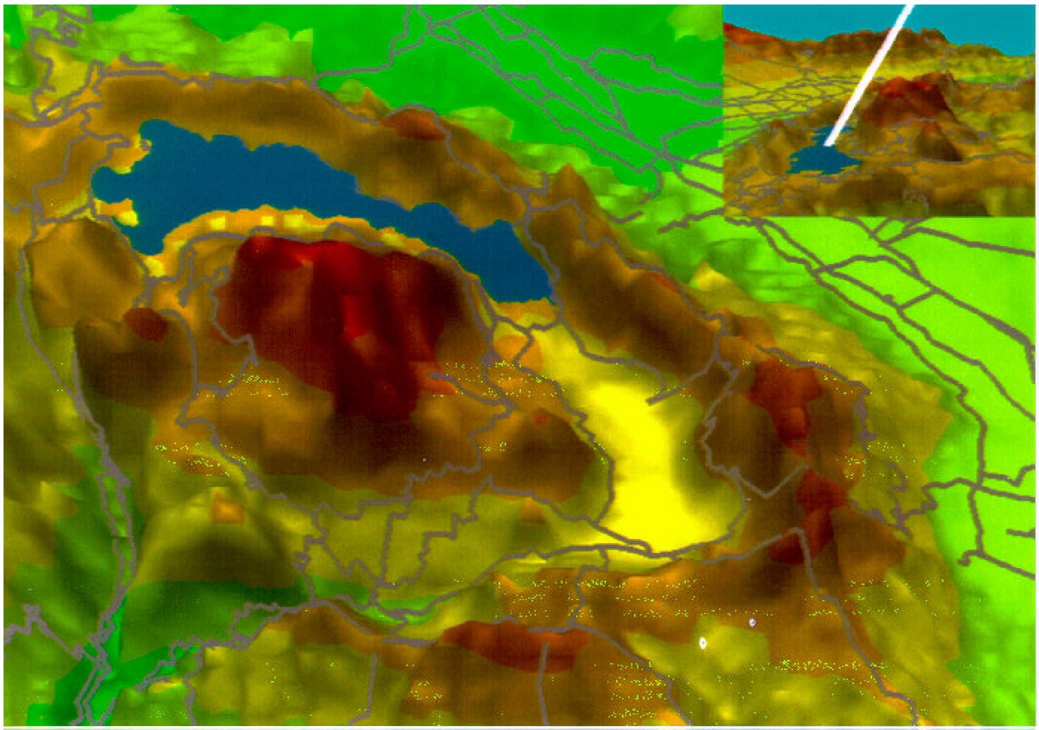
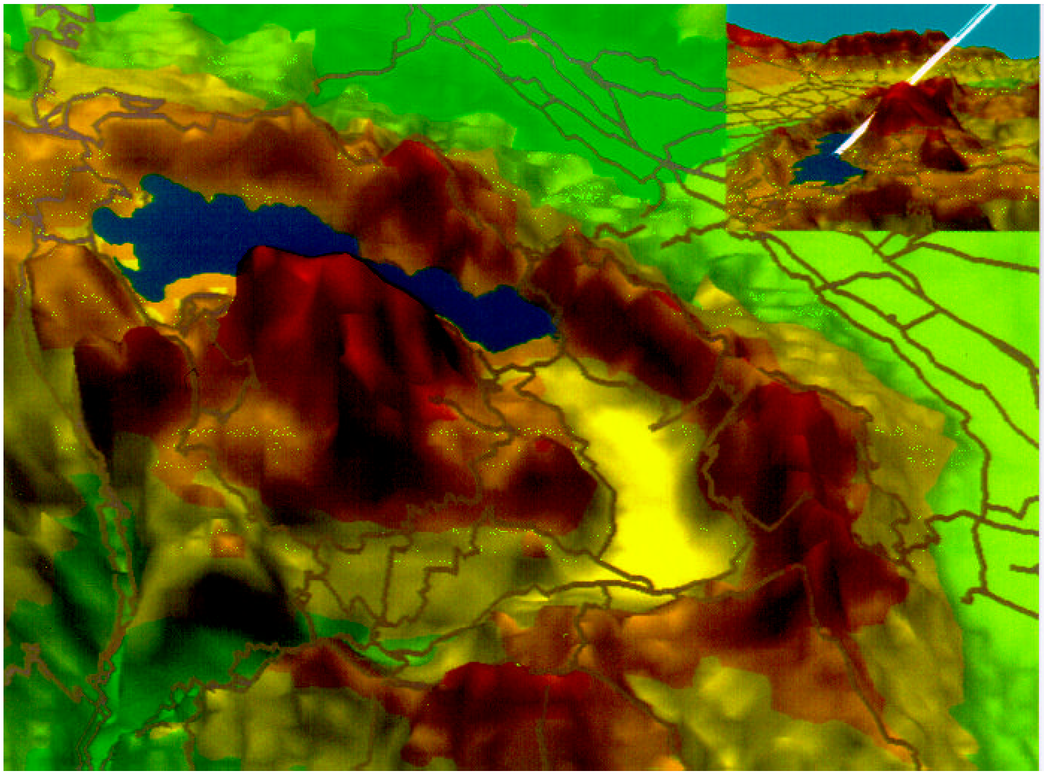


Figure 8: A View of Lake Ashinoko as Seen from the Top of Mt. Kamiyama.

As a typical application of manifolds we take an area guide map generation. A guide map is a sheet of paper where scenes are pasted together as seen from the vista points appropriate for individual sceneries [7], [8].



. Figure 9: A View of Mt. Kamiyama as Seen from the Opposite Side of Lake Ashinoko.

Figure 10 gives a case of a guide of a part of this famous scenic area, Hakone, pasting the two vi Figure 8 is a view of Lake Ashinoko as seen from the top of Mt. Kamiyama and Figure 9 is another view, the view of Mt. Kamiyama as seen from the opposite side of Lake Ashinoko. Figure 10 is a case of a guide map obtained pasting the two views and the views of the neighboring. The cellular overlaps for gluing are normalized by the partition of unity.

We have been studying further applications of cellular structured spaces. However, in designing cyberspaces that are quite general, usually we cannot have the luxury of enjoying such a limited space built on top of such assumptions. The most of the real world applications have singularities, and non-diffeomorphic cellular spaces. Hence, we usually use filtration spaces to model both virtual and real worlds in cyberspaces.

The research in cellular structured spaces in technological applications has to go through careful case studies and investigations that will also require significant amount of basic research in mathematics. There is a significant problem in developing curriculums and courseware to educate and train students and technological professionals in all application domains.

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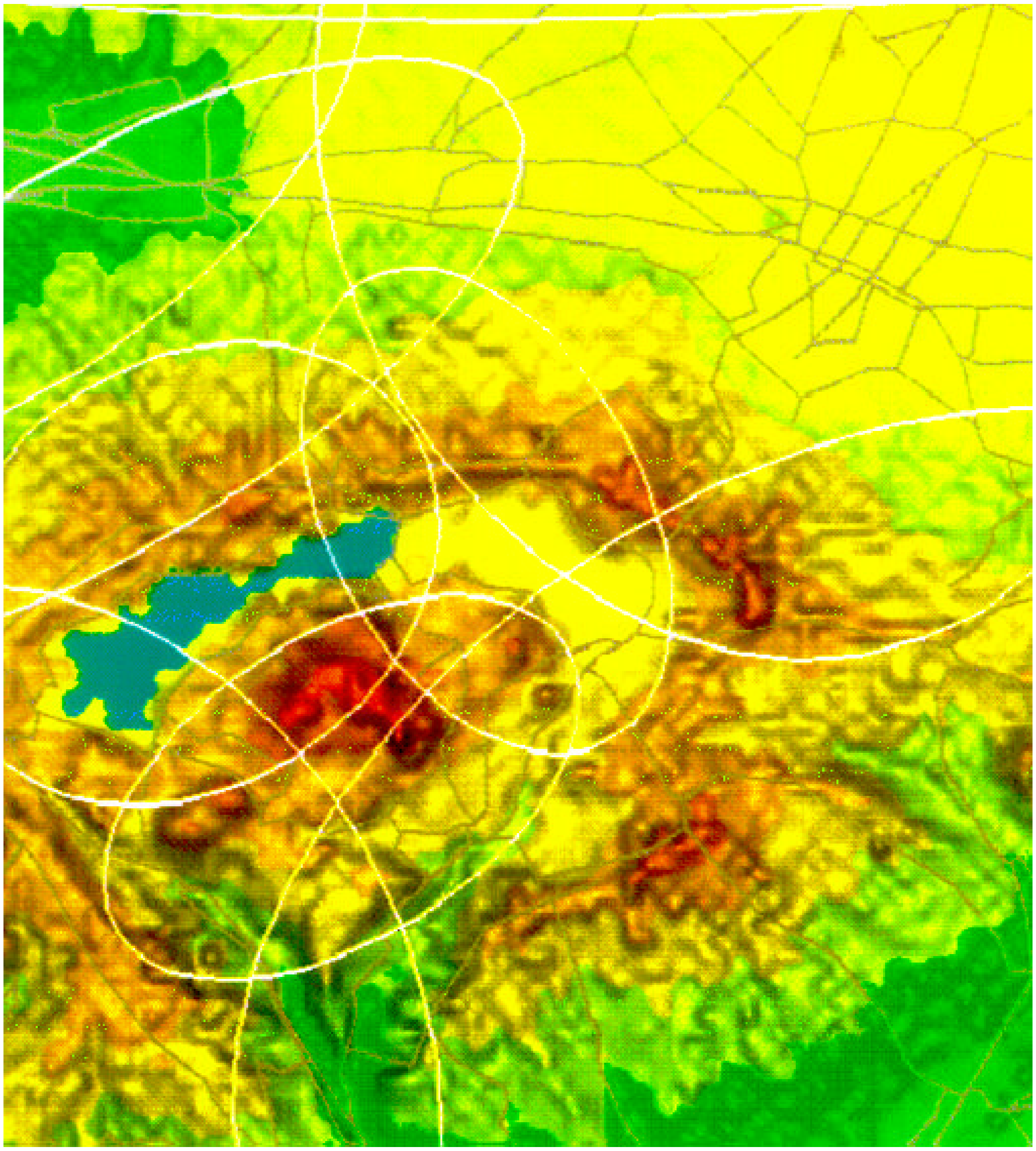

Figure 10: An Example of a Guide Map of Lake Ashinoko and Mt. Kamiyama Area in the Hakone Scenic Sites.

6 Epilogue

A filtration is an extension of the concept of a *poset* (a partially ordered set); a poset is itself an extension of the concept of a *graph*. Even in the most practiced technological world, set theory as seen in the Z system for software design specification, graph theory in the entity-relationship data models, and posets in communication network design are still among the advanced approaches. Such very significant delay in practicing mathematics in technology justifies the need to promote ATCM more widely and internationally. Further, to promote new mathematical directions, technological requirements are expected to work to axiomatize and theorize the computational frameworks of information technology.

We showed that designing cyberspaces, because of their extreme complexity, requires us to use far more general modeling approach than those widely used so far such that we can employ powerful mathematical knowledge on networked computers based on cellular spatial structures. People can work locally within each cell instead of getting their views scattered by the global world; cellular spatial structures provide enough assurance to get the cells put together for global integration in a well designed manner. In this era of interconnecting all local existences globally through information networks, such cellular structured spaces play essential roles.

After over a decade of intensive work, the research reported here is still at the beginning.

7 Acknowledgements

The research has been partially supported by Hosei University. Recent support is also coming from Monolith, Fukushima Prefecture and Ricoh Software Research Center. The effort of ATCM98 organizers including Professor Wei-Chi Yang and Professor Tateaki Sasaki to promote this important academic area of “technology of mathematics” is greatly appreciated.

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